$\top$-AVOIDING RECTANGULATIONS, INVERSION SEQUENCES, AND DYCK PATHS

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joint work with Andrei Asinowski[2] (Alpen-Adria-Universität Klagenfurt)

Permutation Patterns 2024
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[2] Supported by FWF – Austrian Science Fund

Definitions and Terminology

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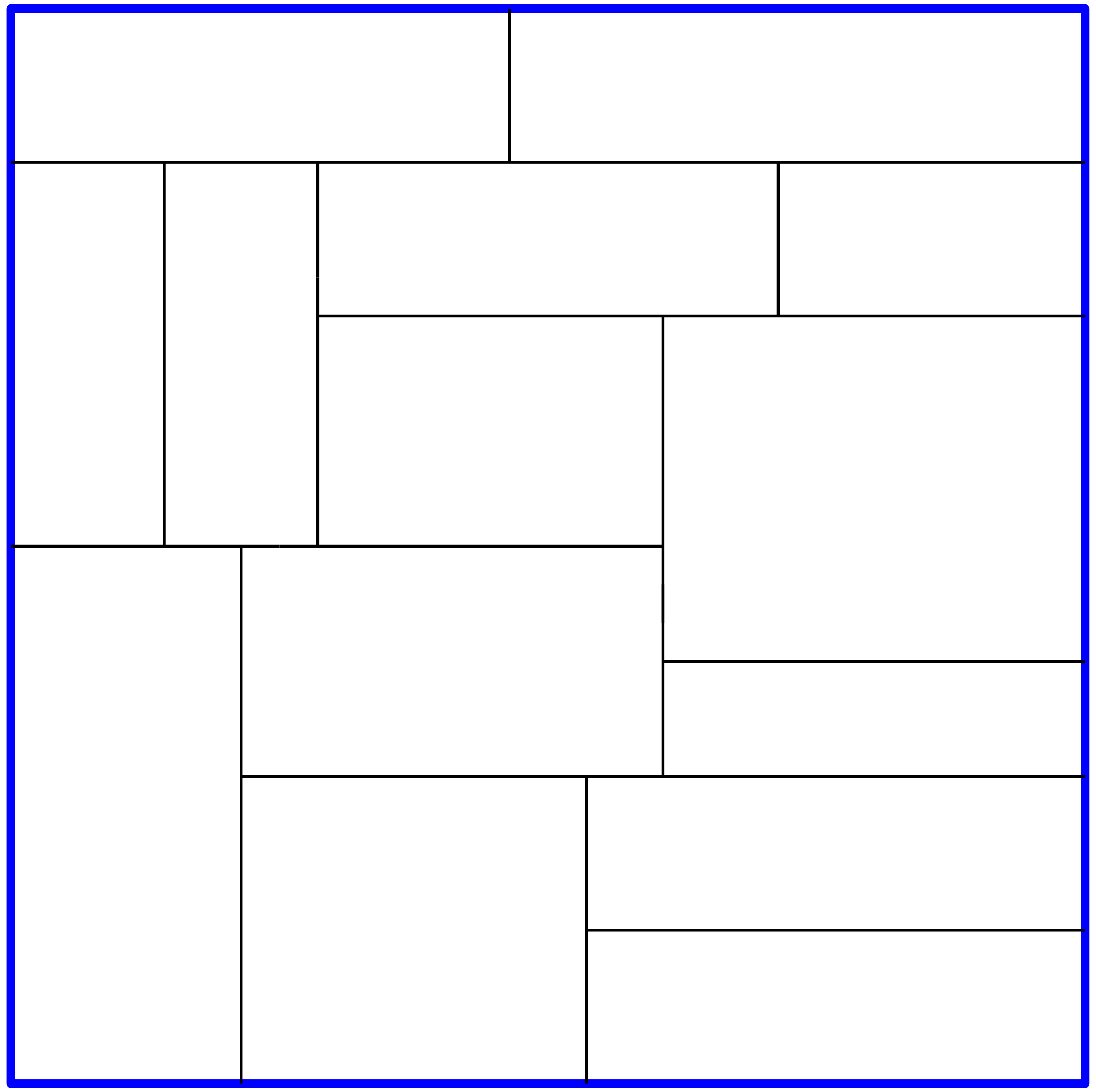

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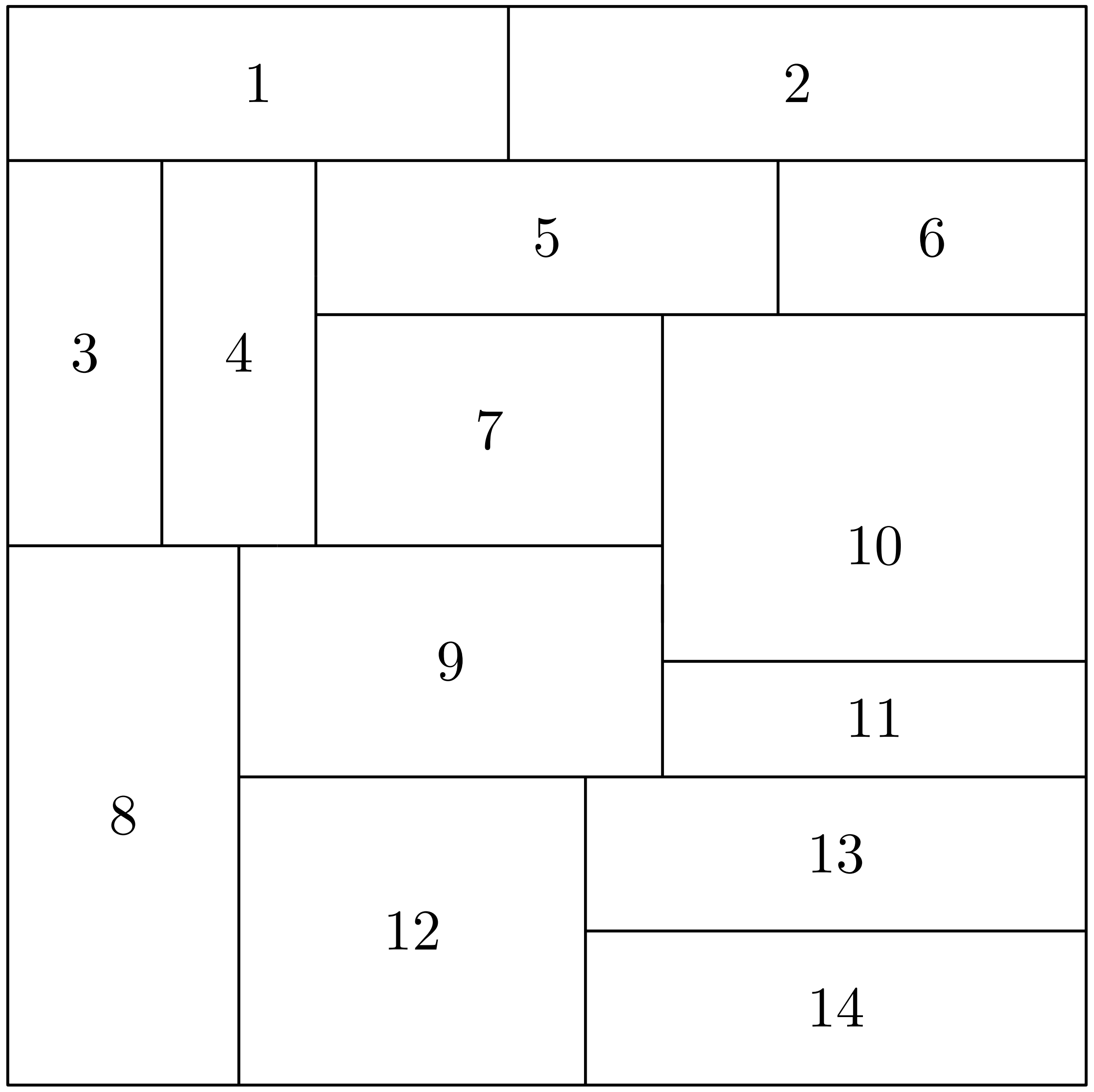

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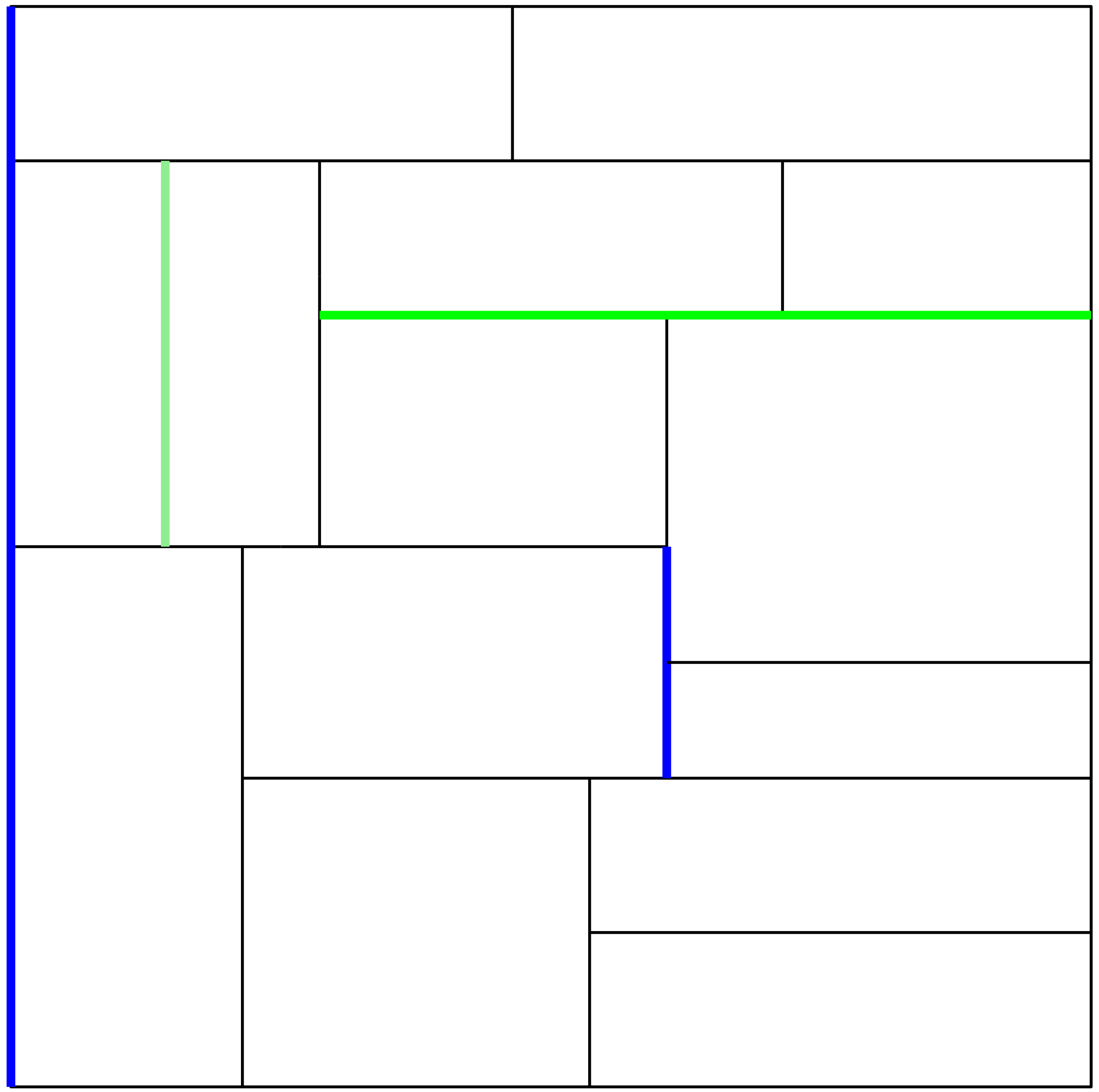

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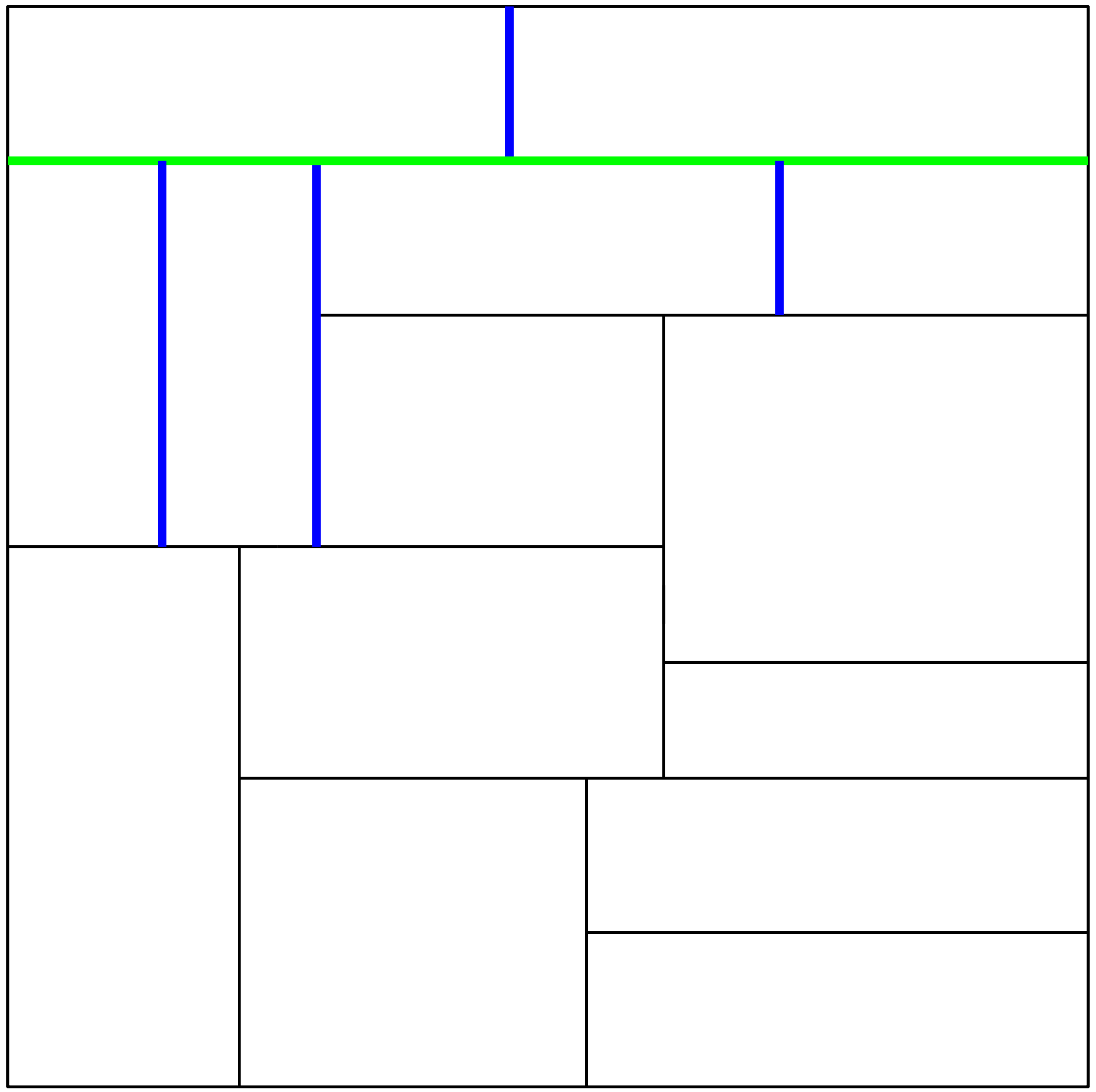

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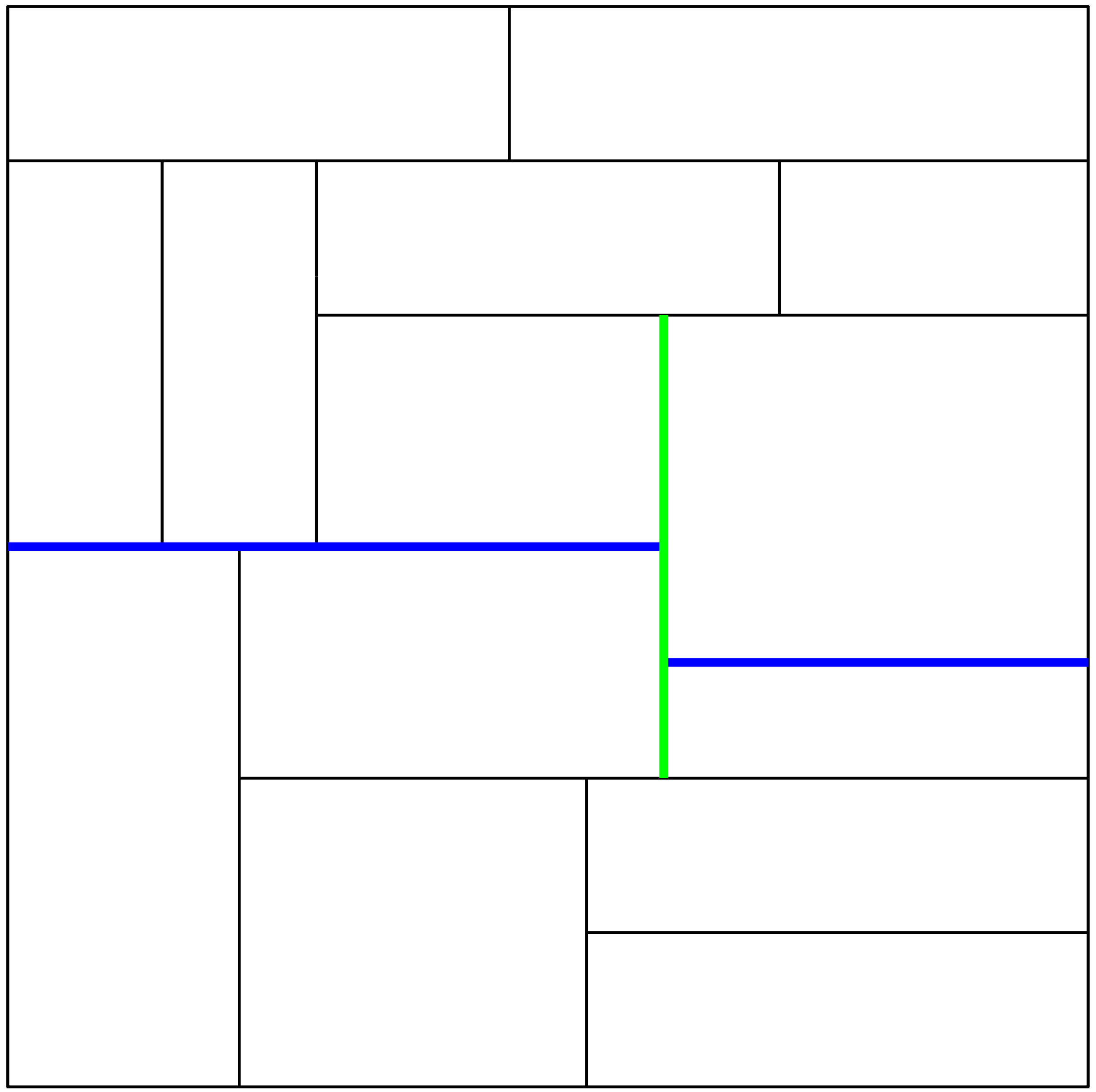

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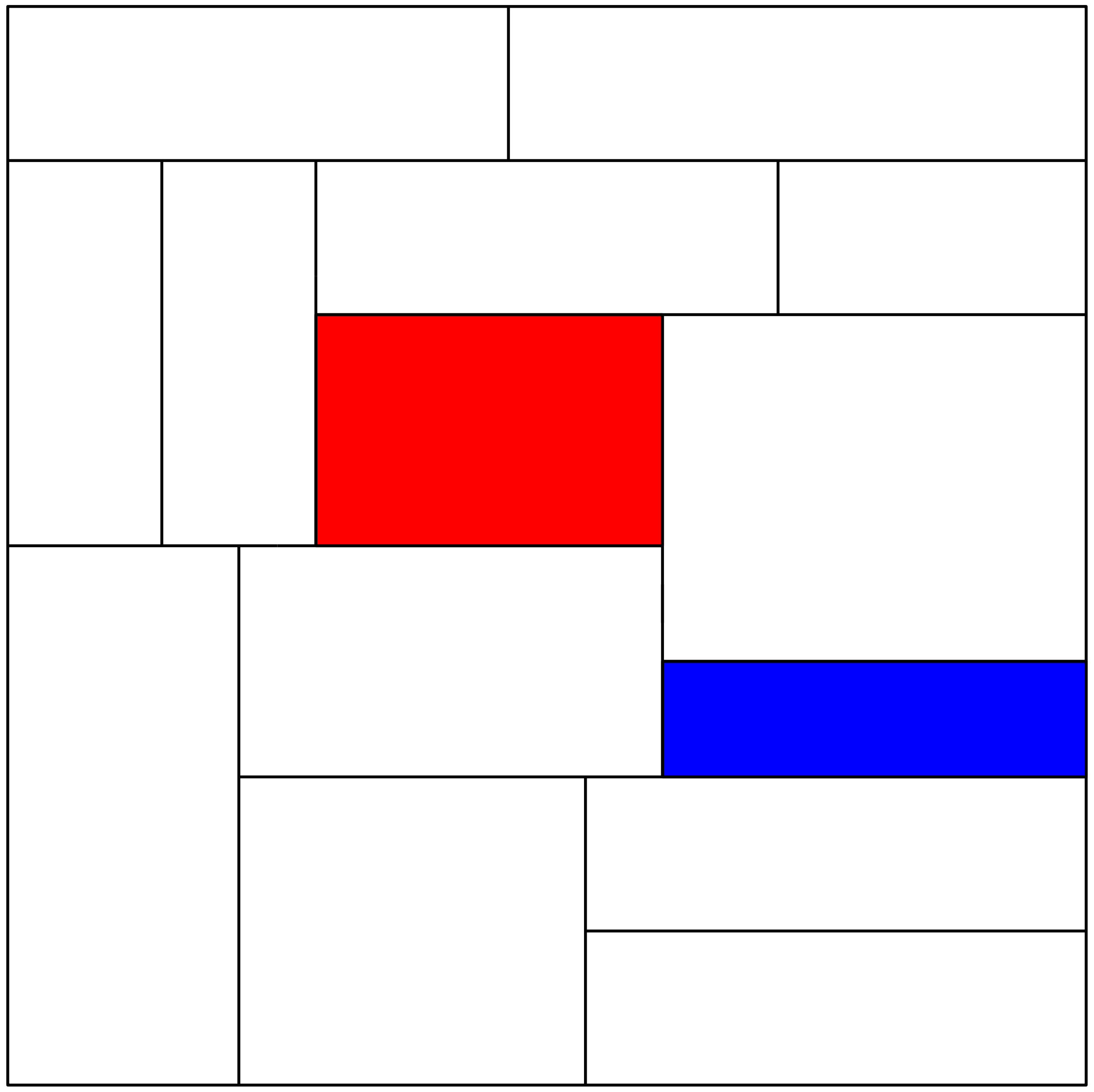
Within $\mathcal{R}$, two rectangles are *left/right neighbors* if they are seperated by a single vertical segment.



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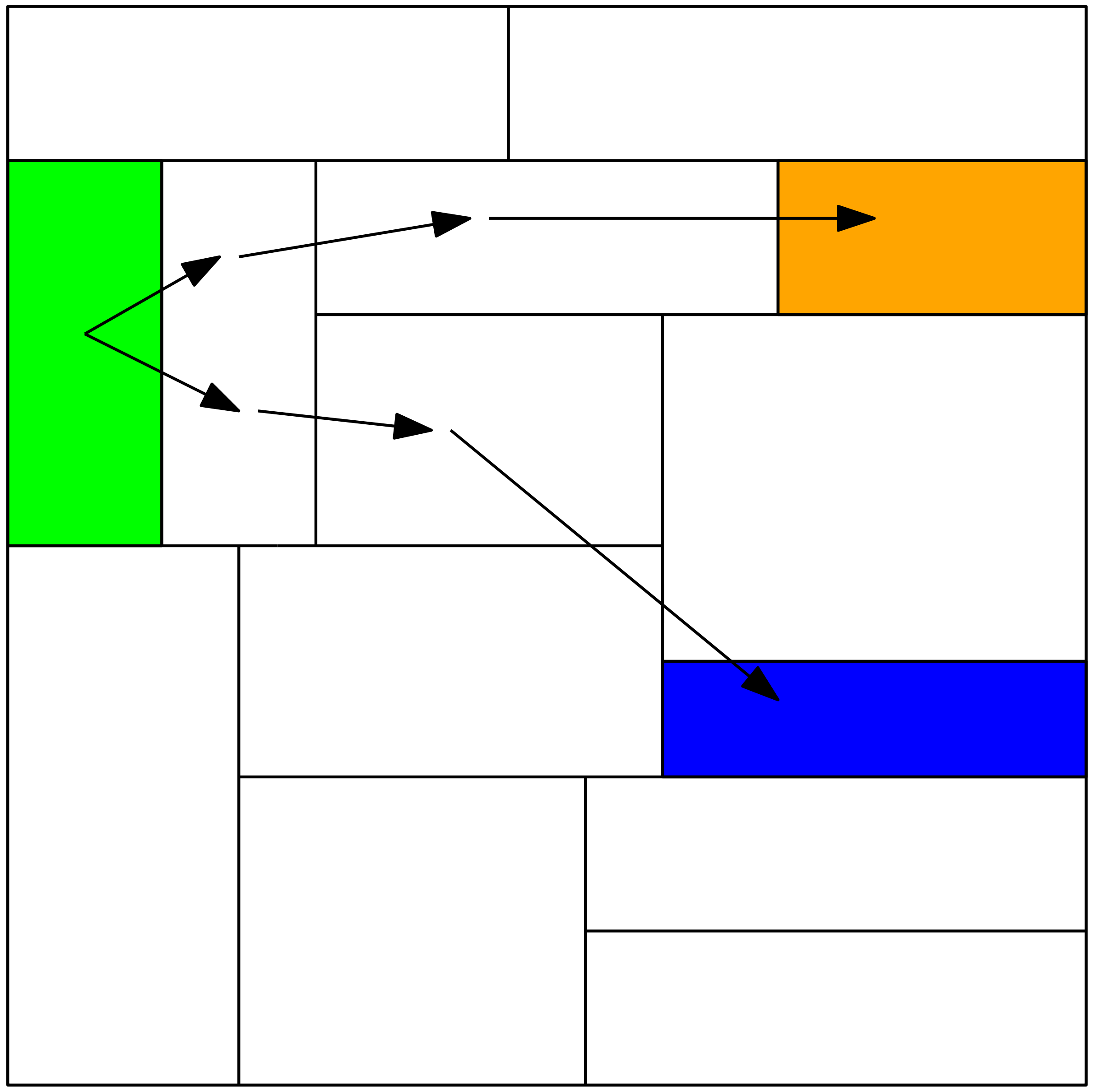
Rectangle r is *left of* rectangle s if they are part of a sequence of left/right neighbors.



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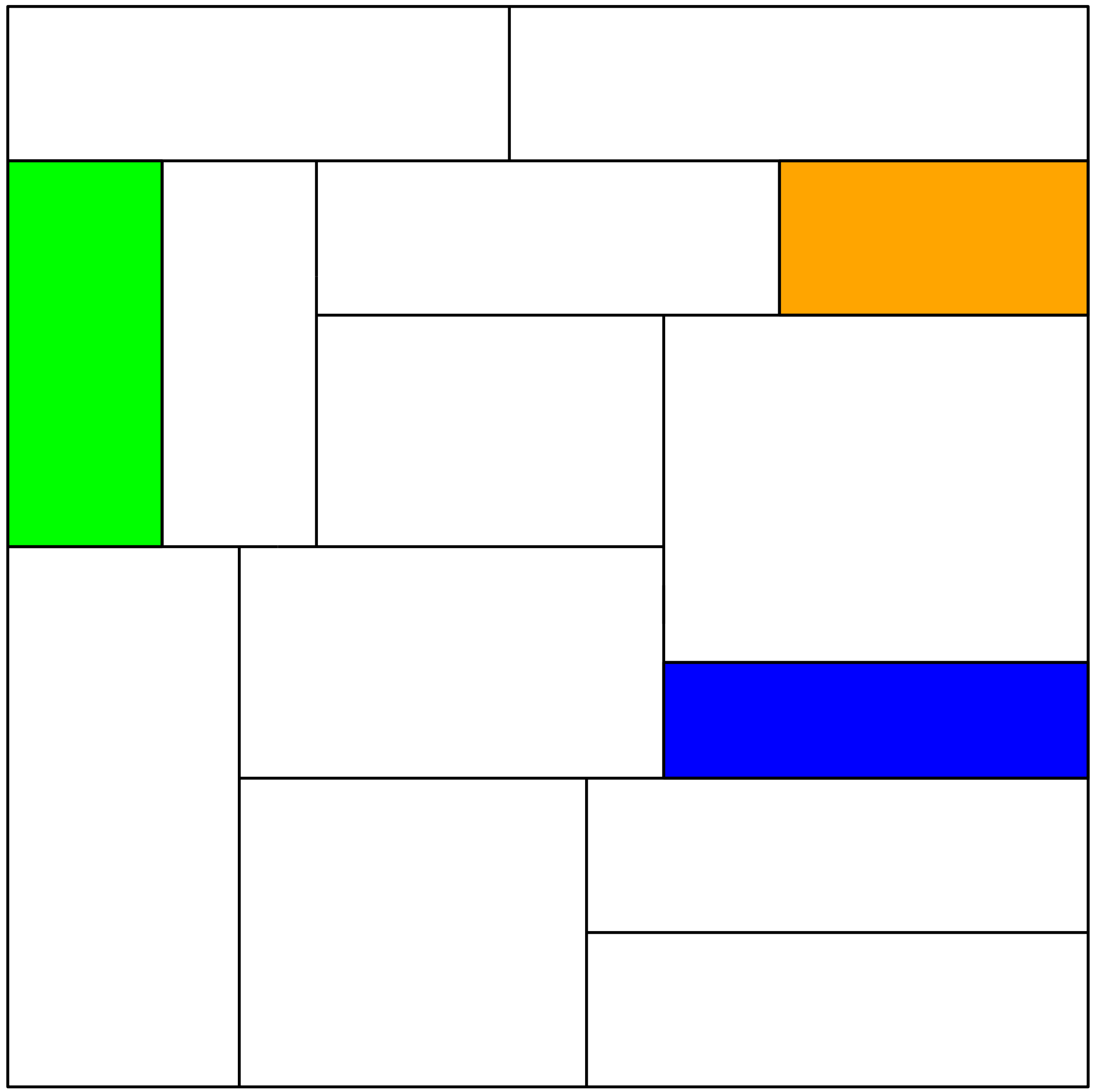

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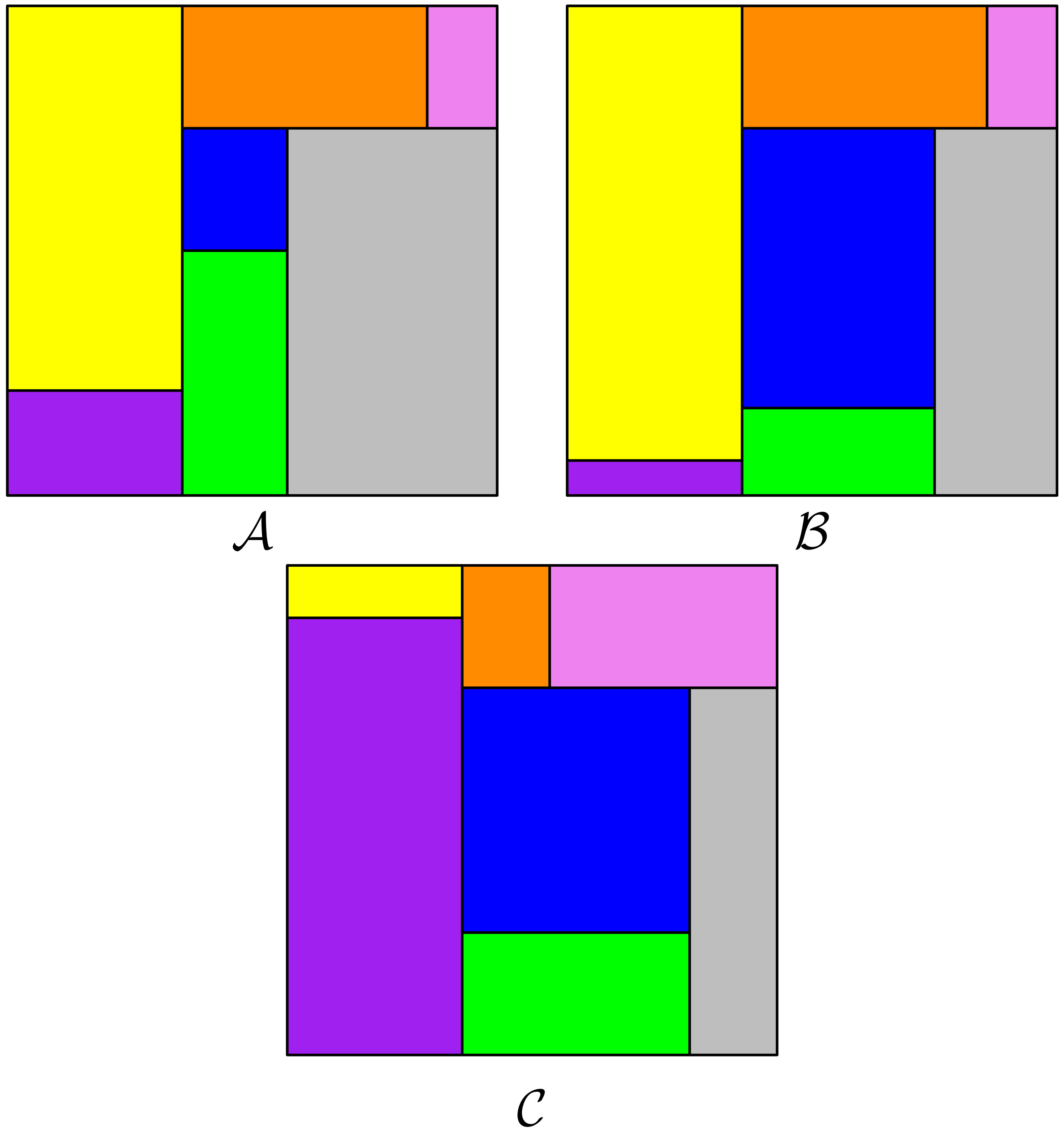

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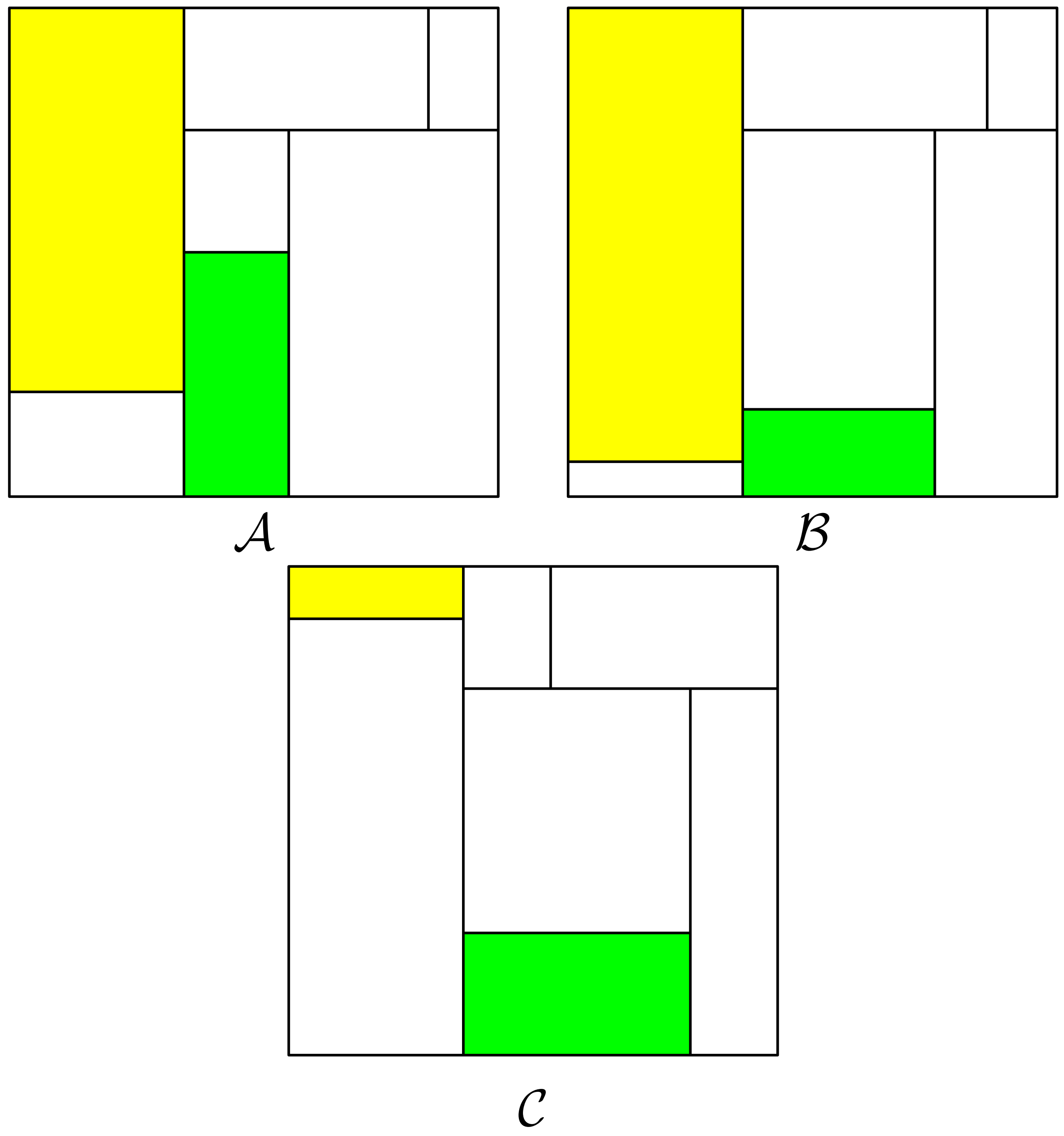

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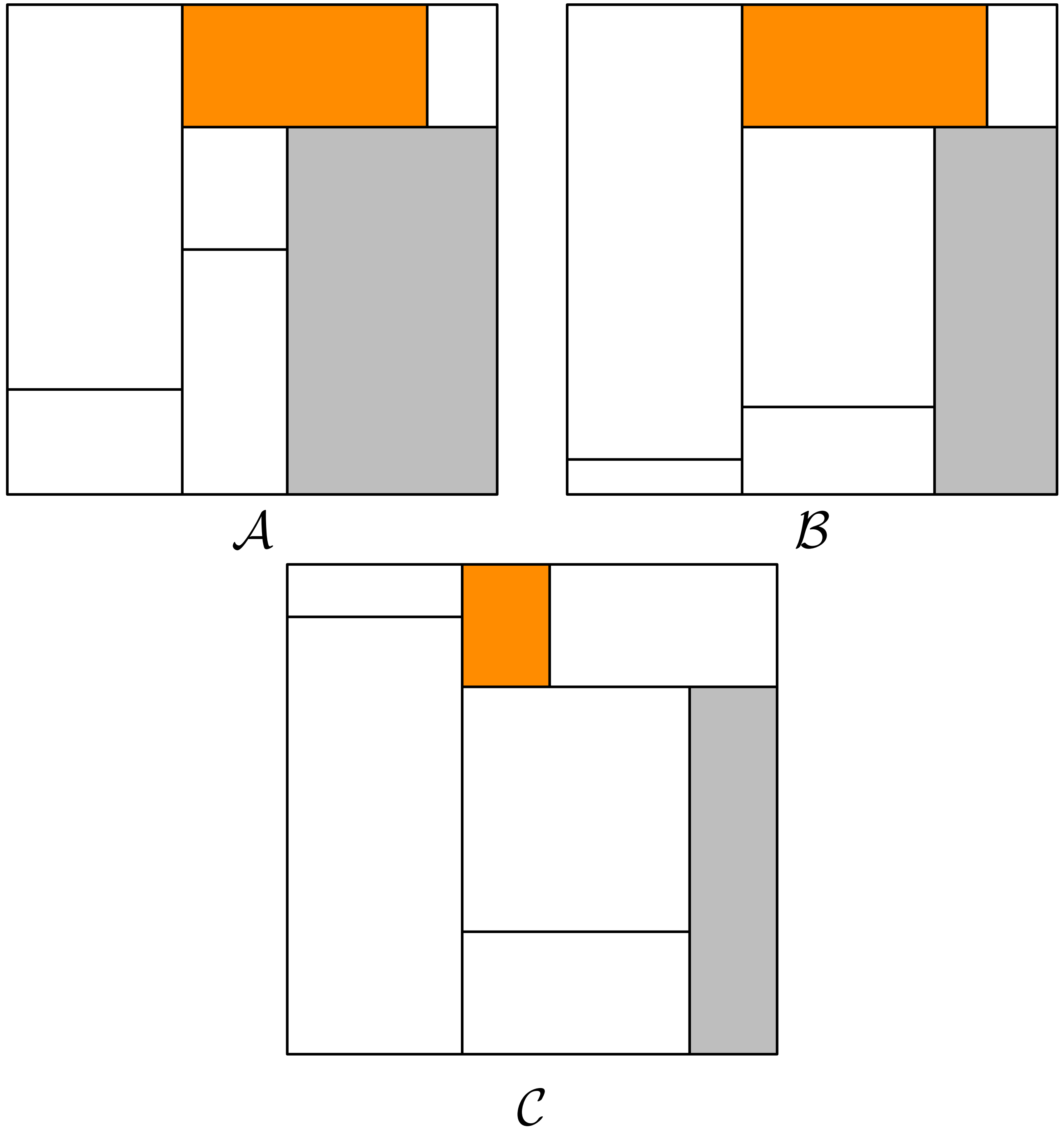

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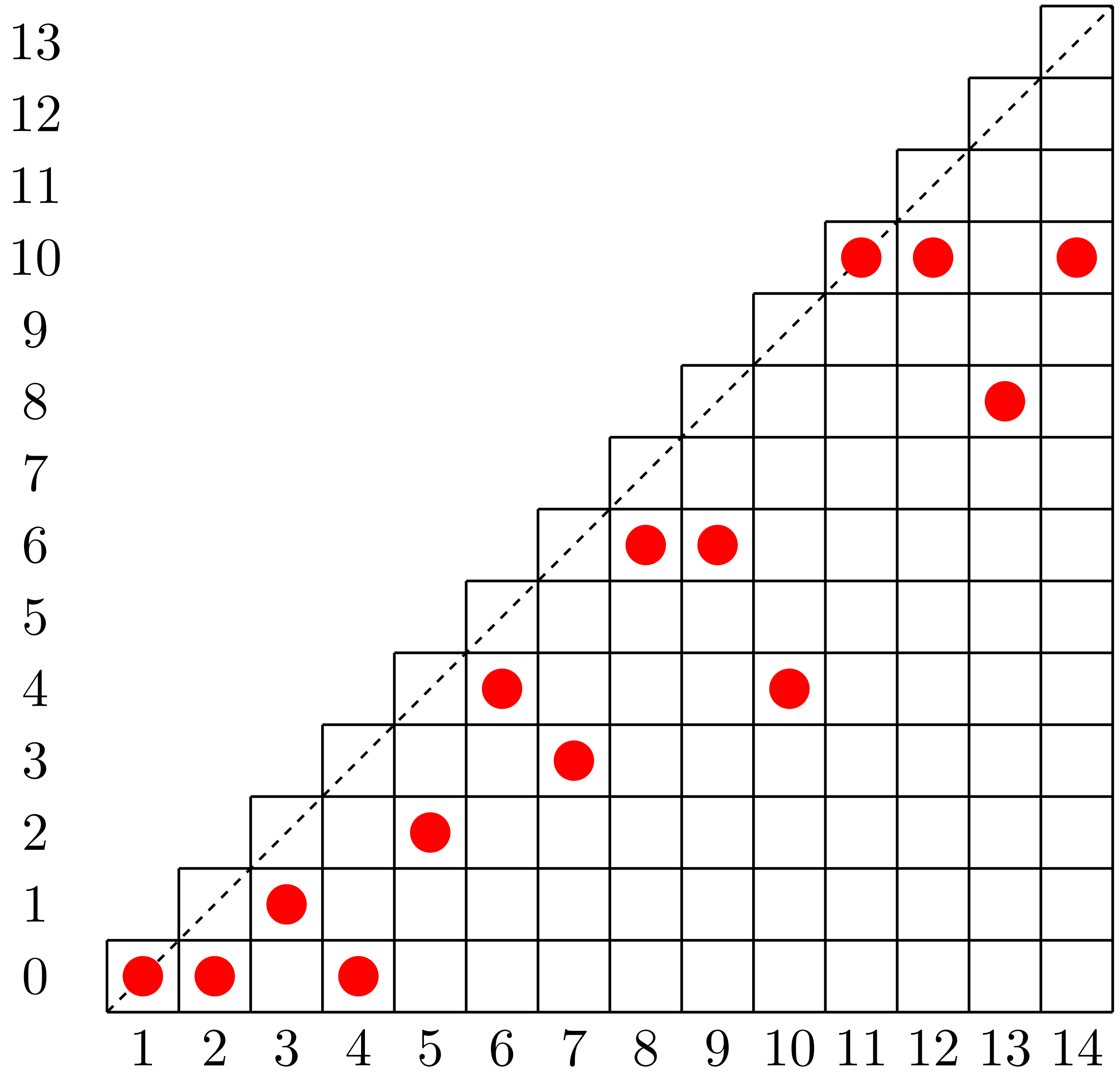
An integer sequence $s = (s_1, s_2, \ldots, s_n)$ is called an *inversion sequence* if for all $1 \leq i \leq n$, we have $0 \leq s_i \leq i - 1$.



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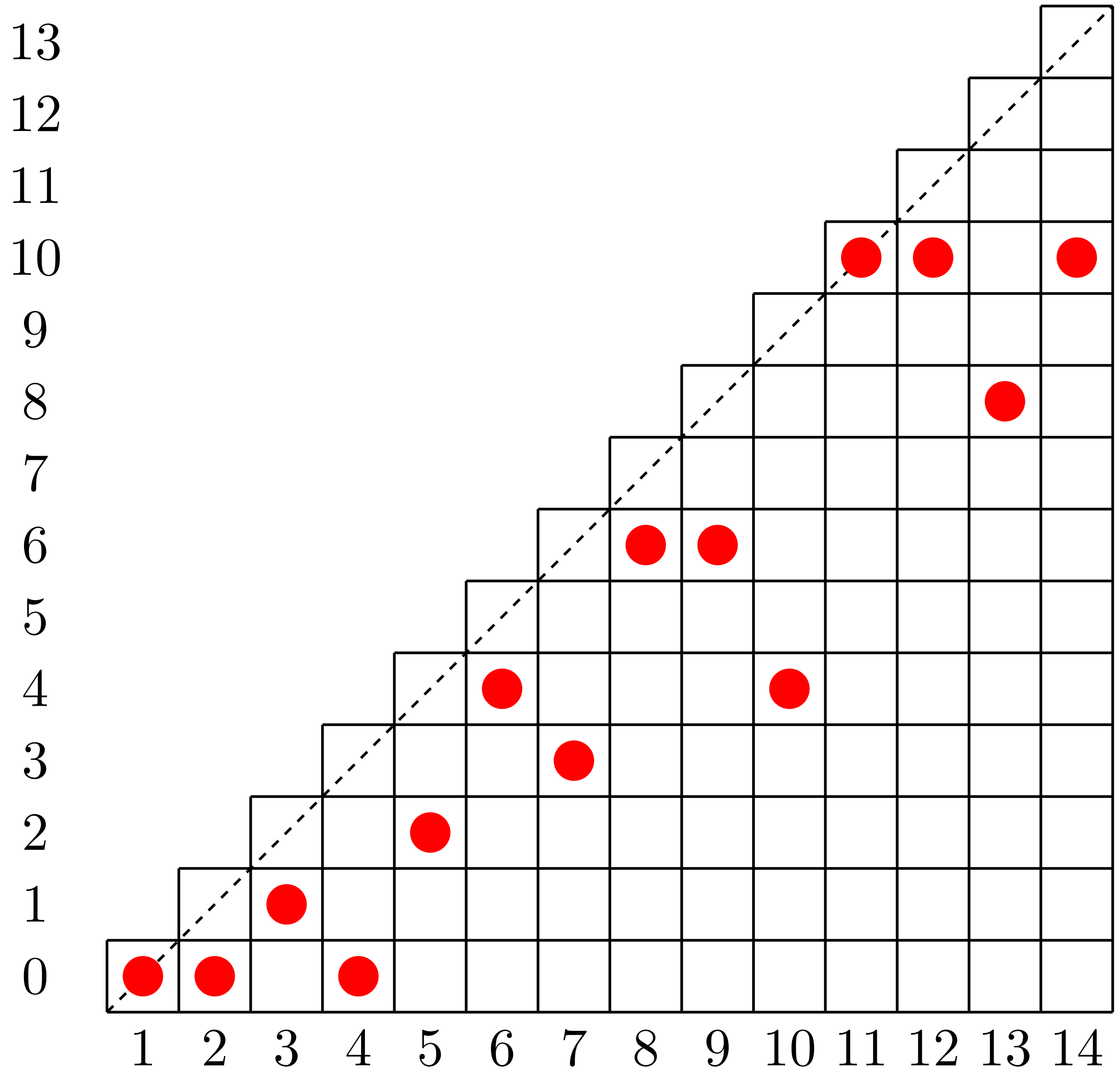
We say s avoids a pattern t if there is no subsequence of s which is order isomophic to t.



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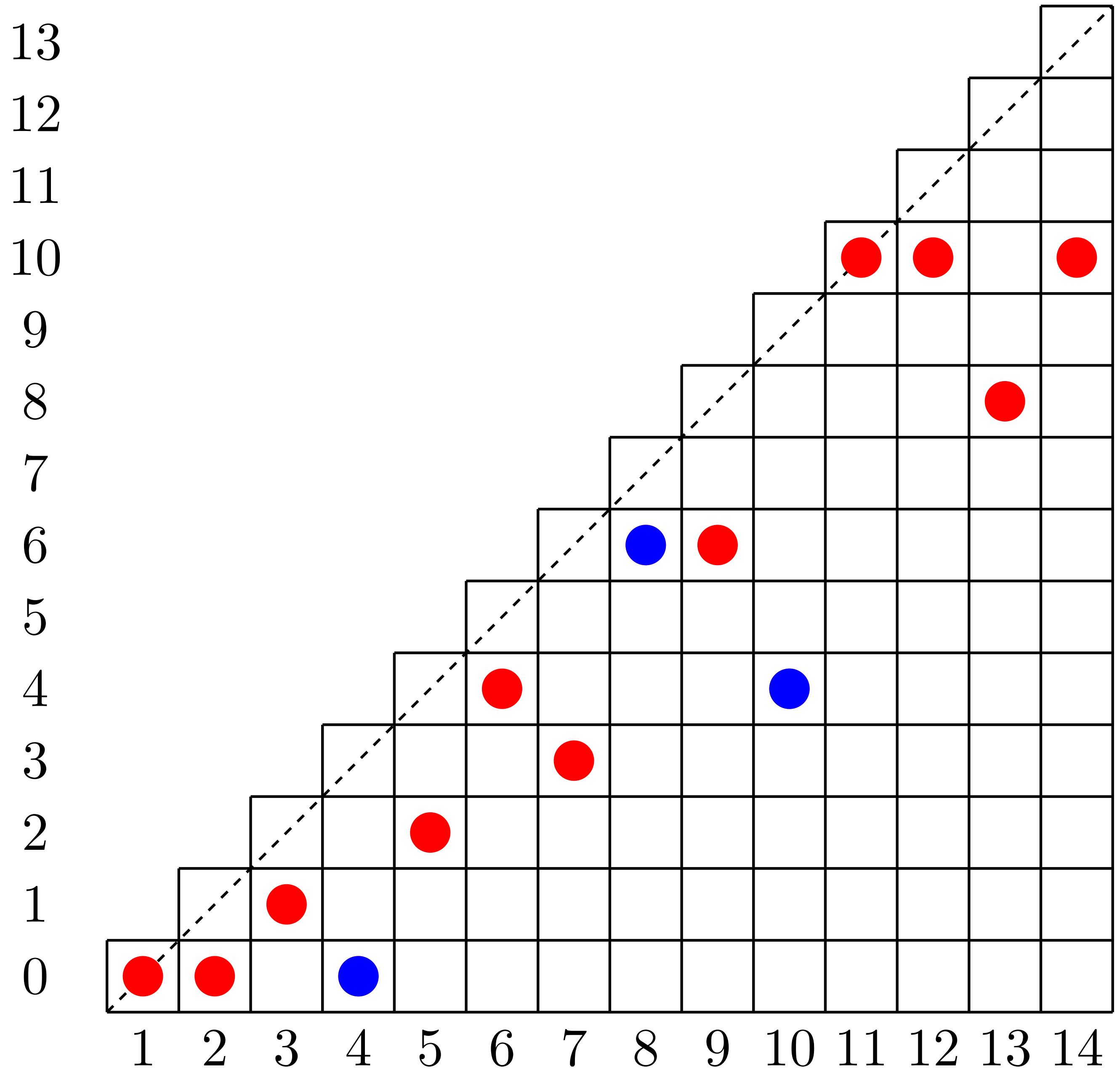

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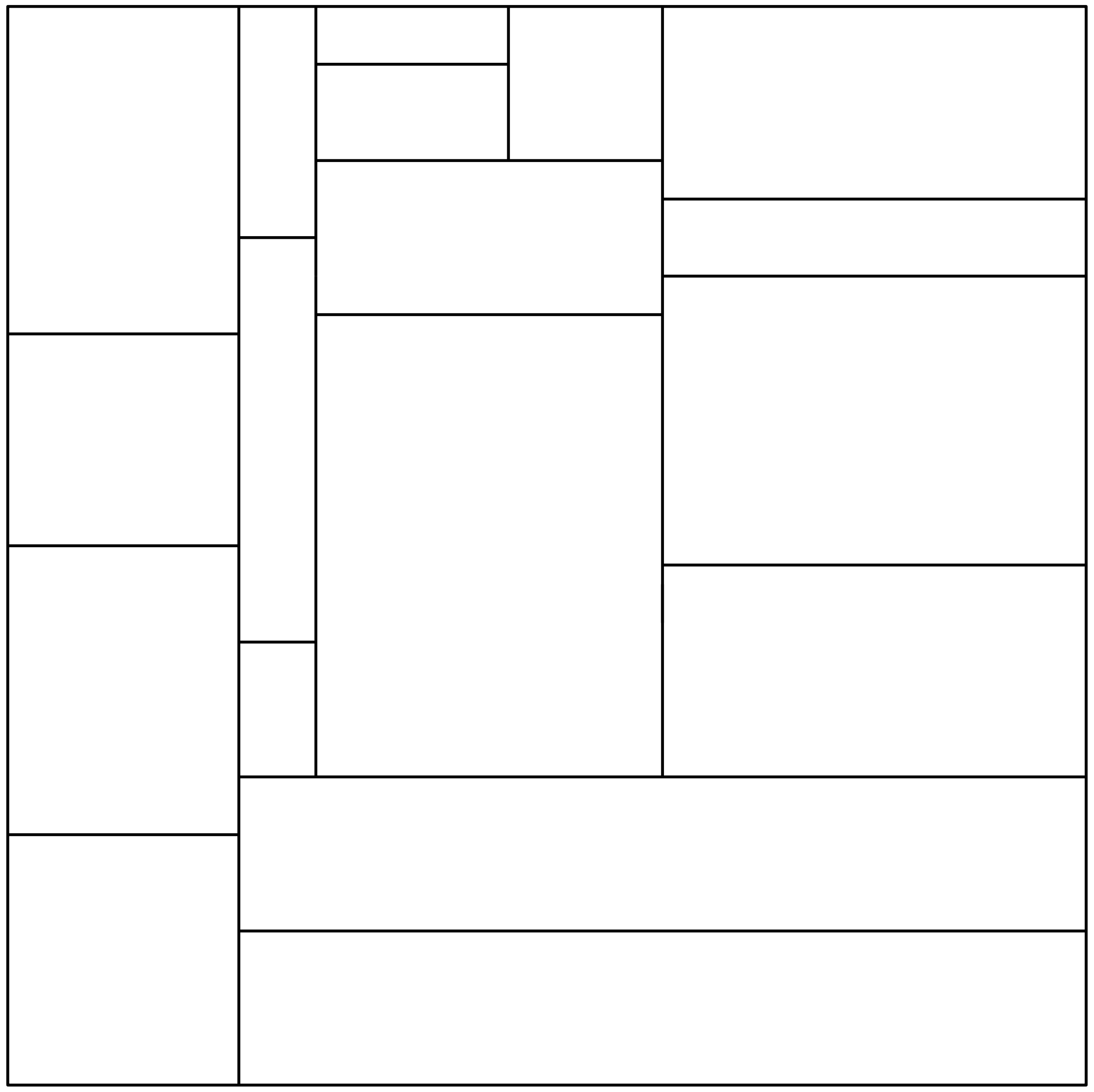

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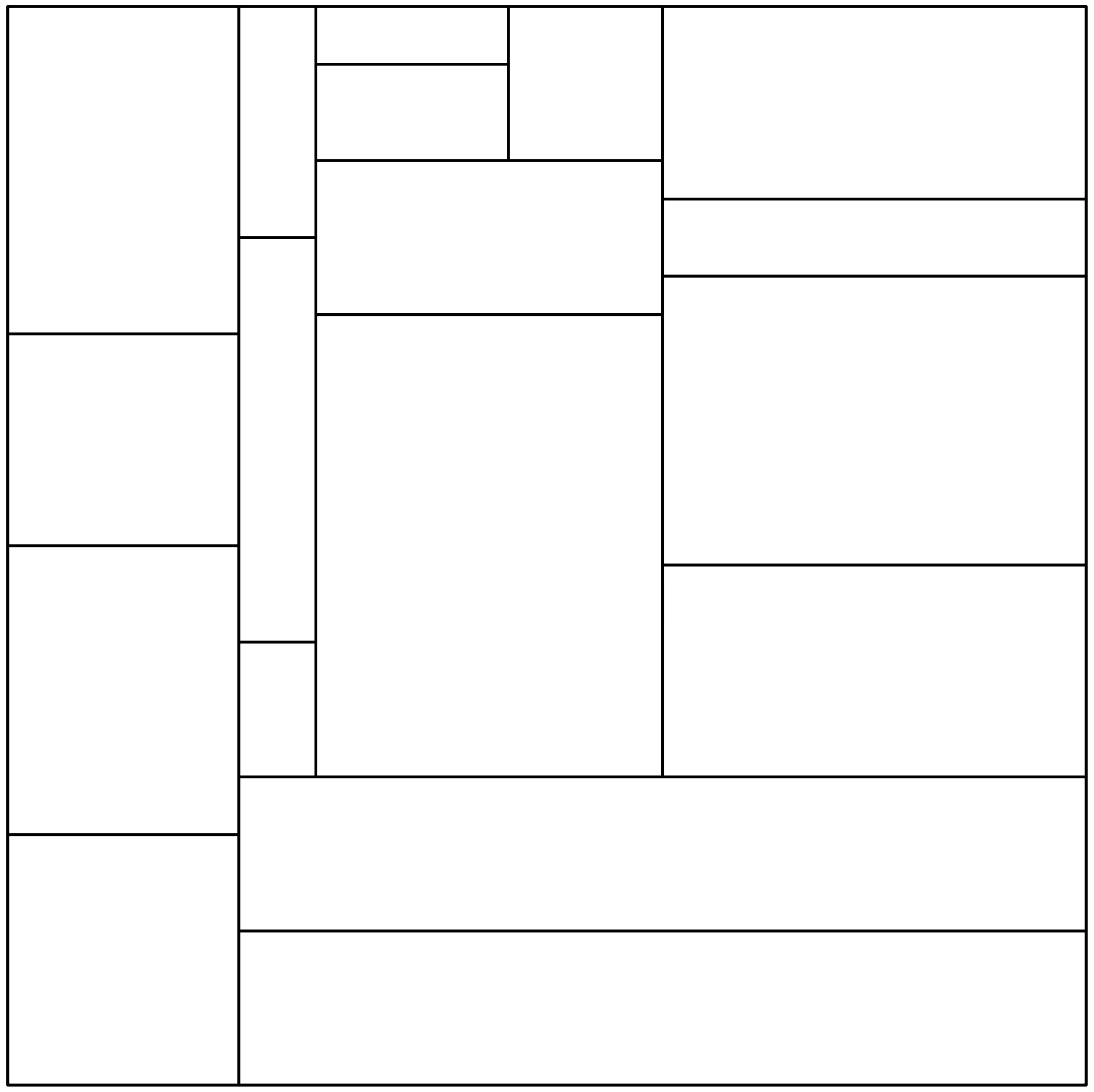
Systematic study of pattern avoidance in rectangulations was started by Merino and Mütze (2021), several models were solved by Asinowski and Banderier (2023).



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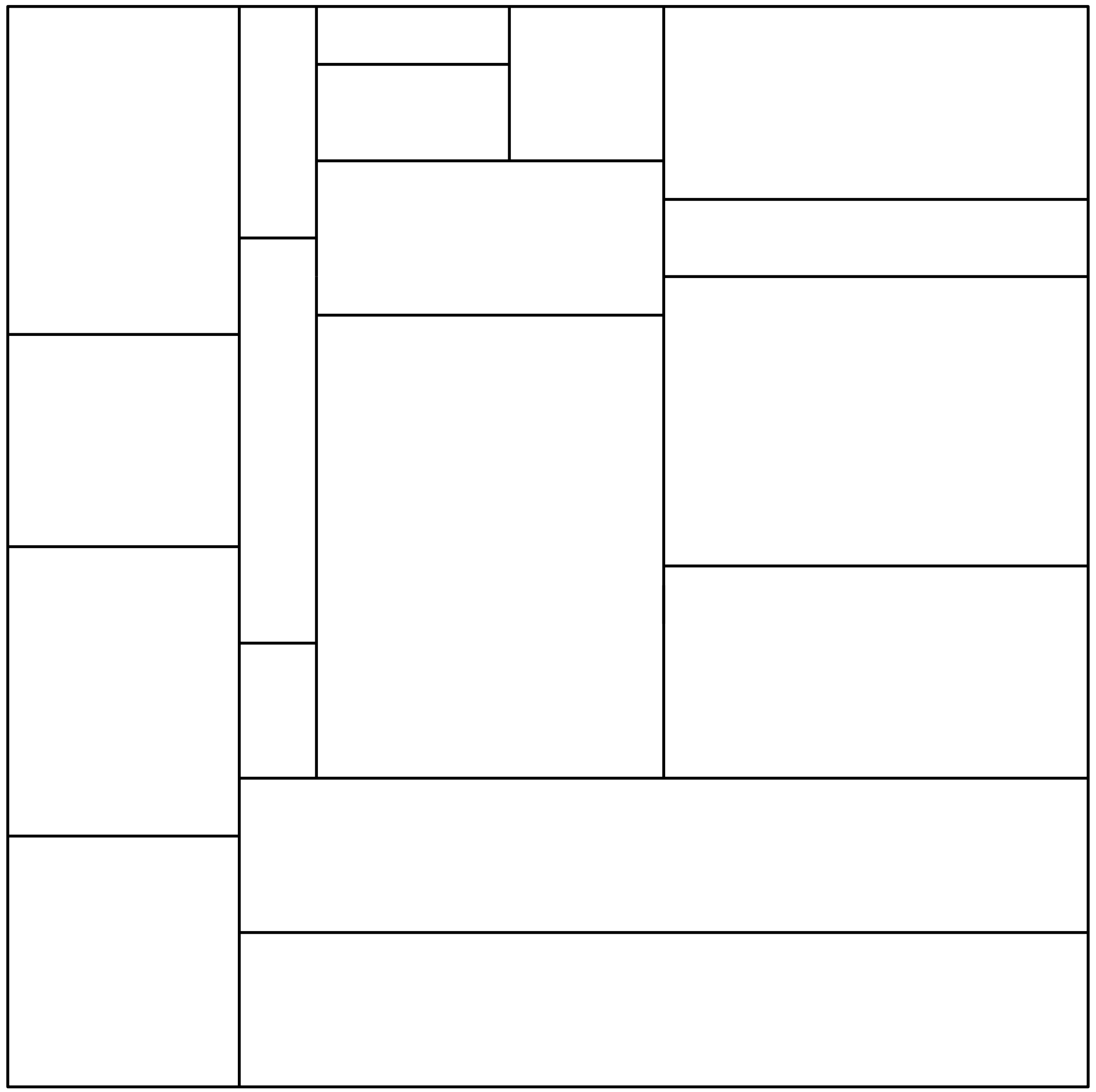
Let L be a set of rectangulation patterns and denote by $R_n^w(L)$ and $R_n^s(L)$ the set of weak and, respectively, strong rectangulations of size n that avoid all patterns in L.

Our results cover all the (essentially different) cases where $L \subseteq \{\top, \perp, \vdash, \dashv\}$.



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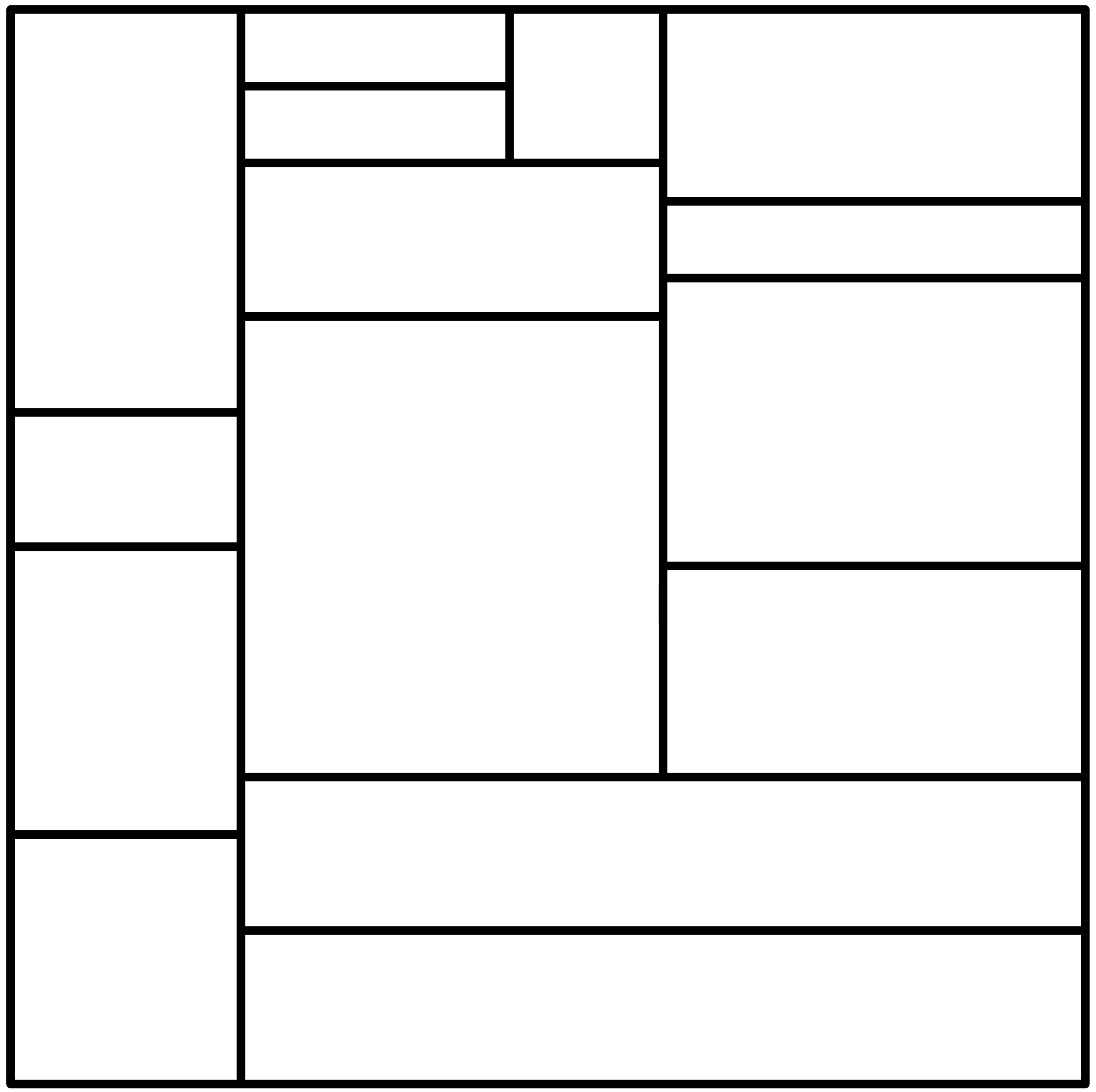
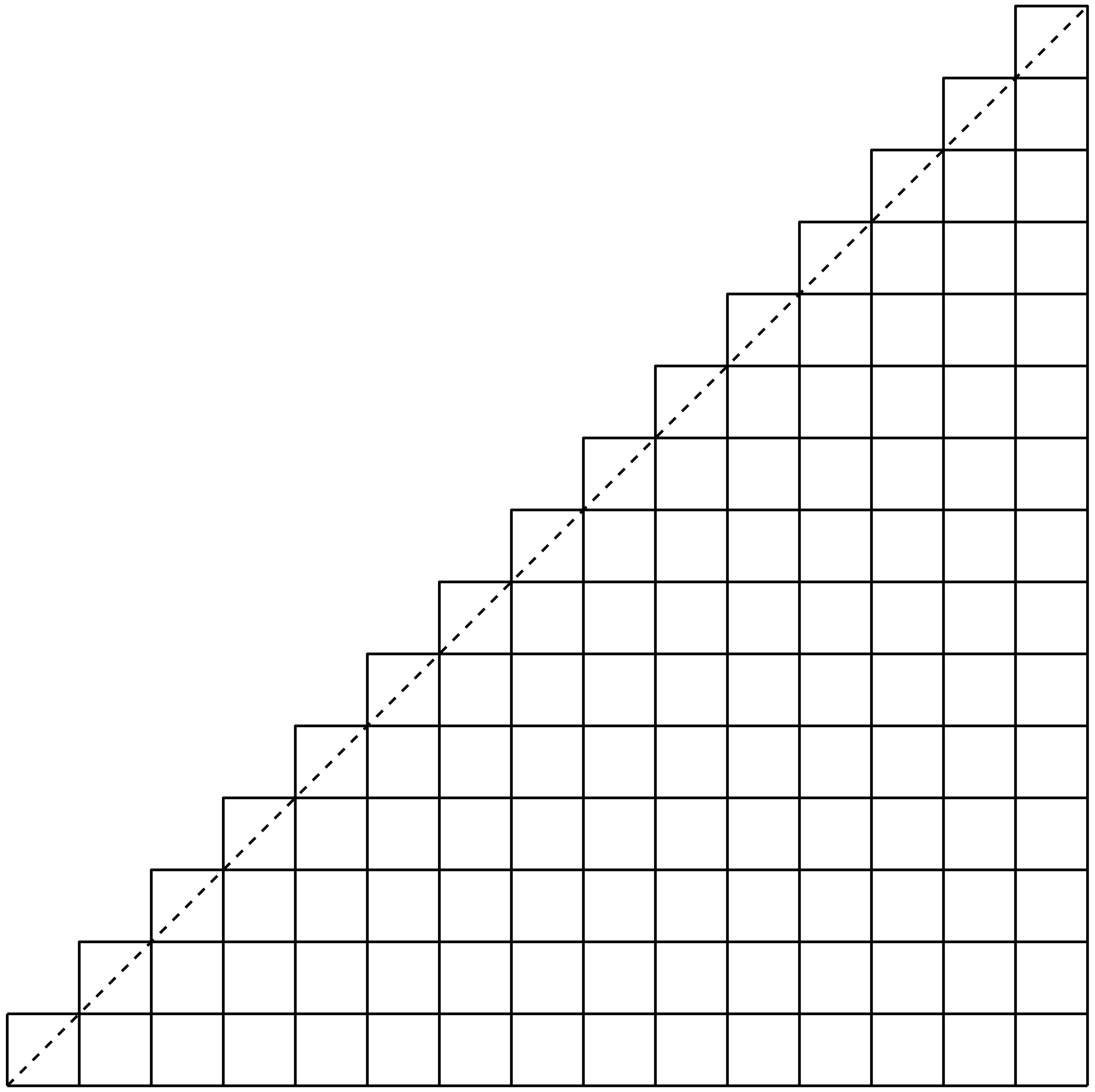
Proof: Bijection to Dyck paths via non-decreasing inversion sequences





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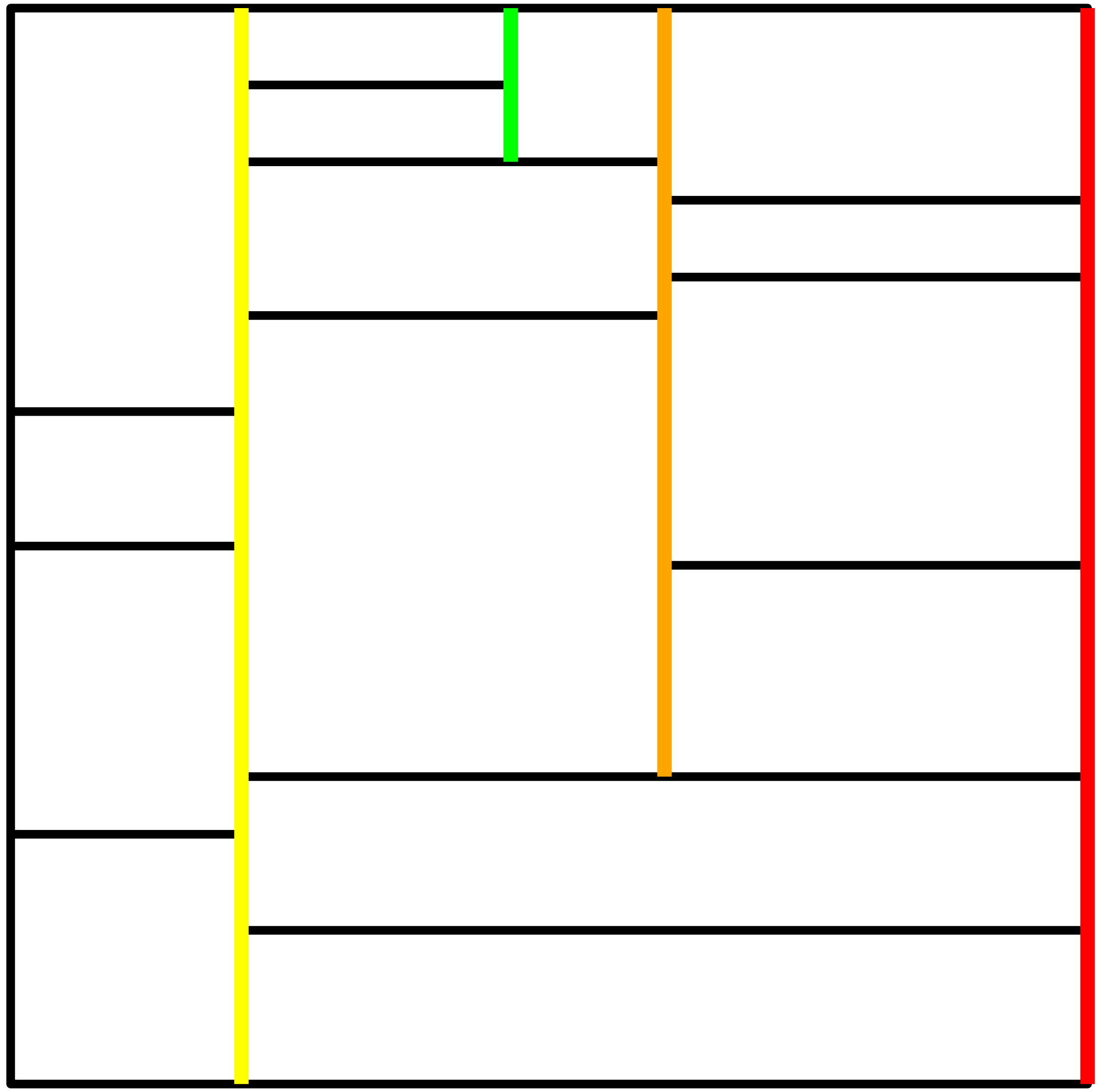

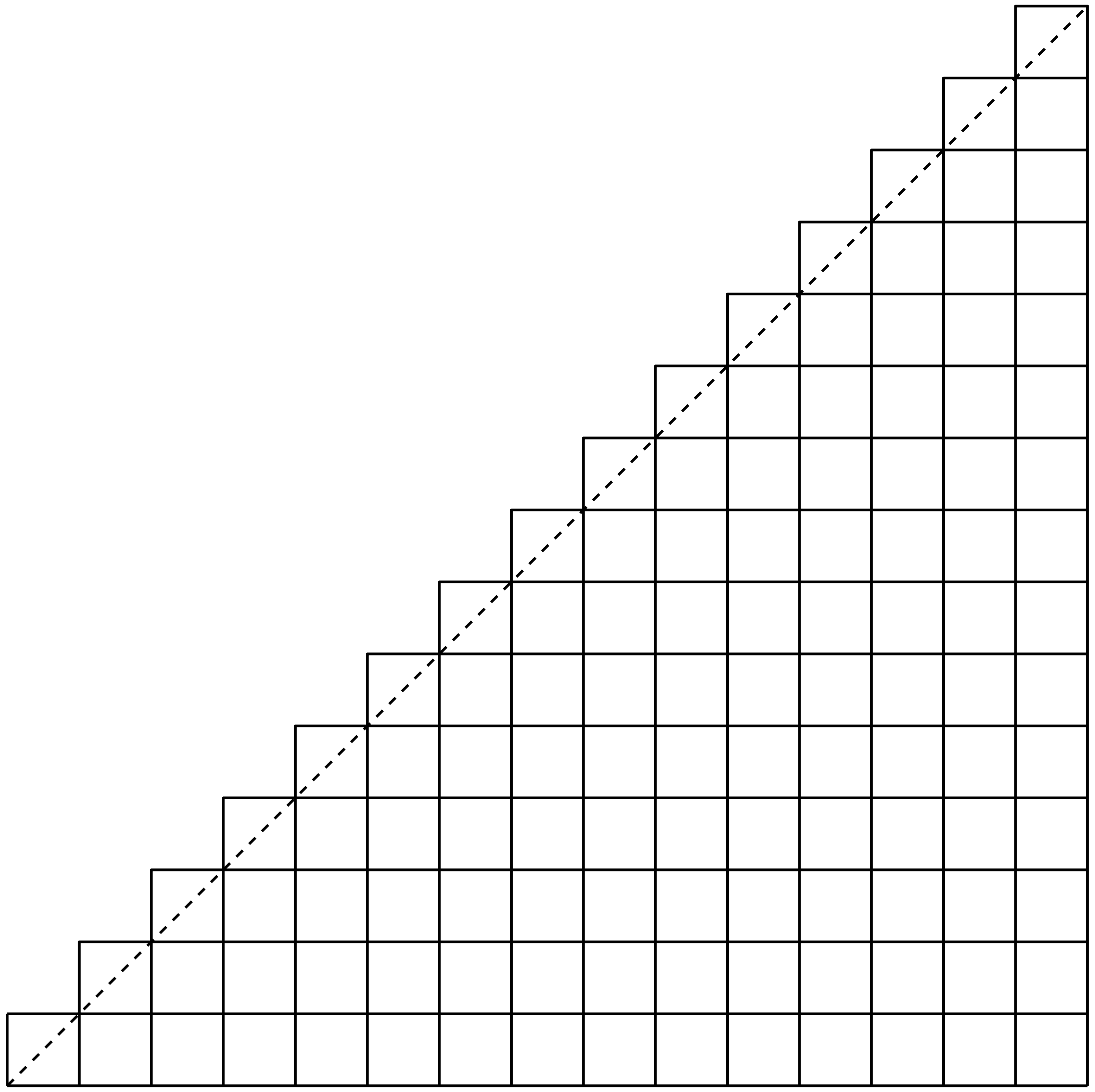

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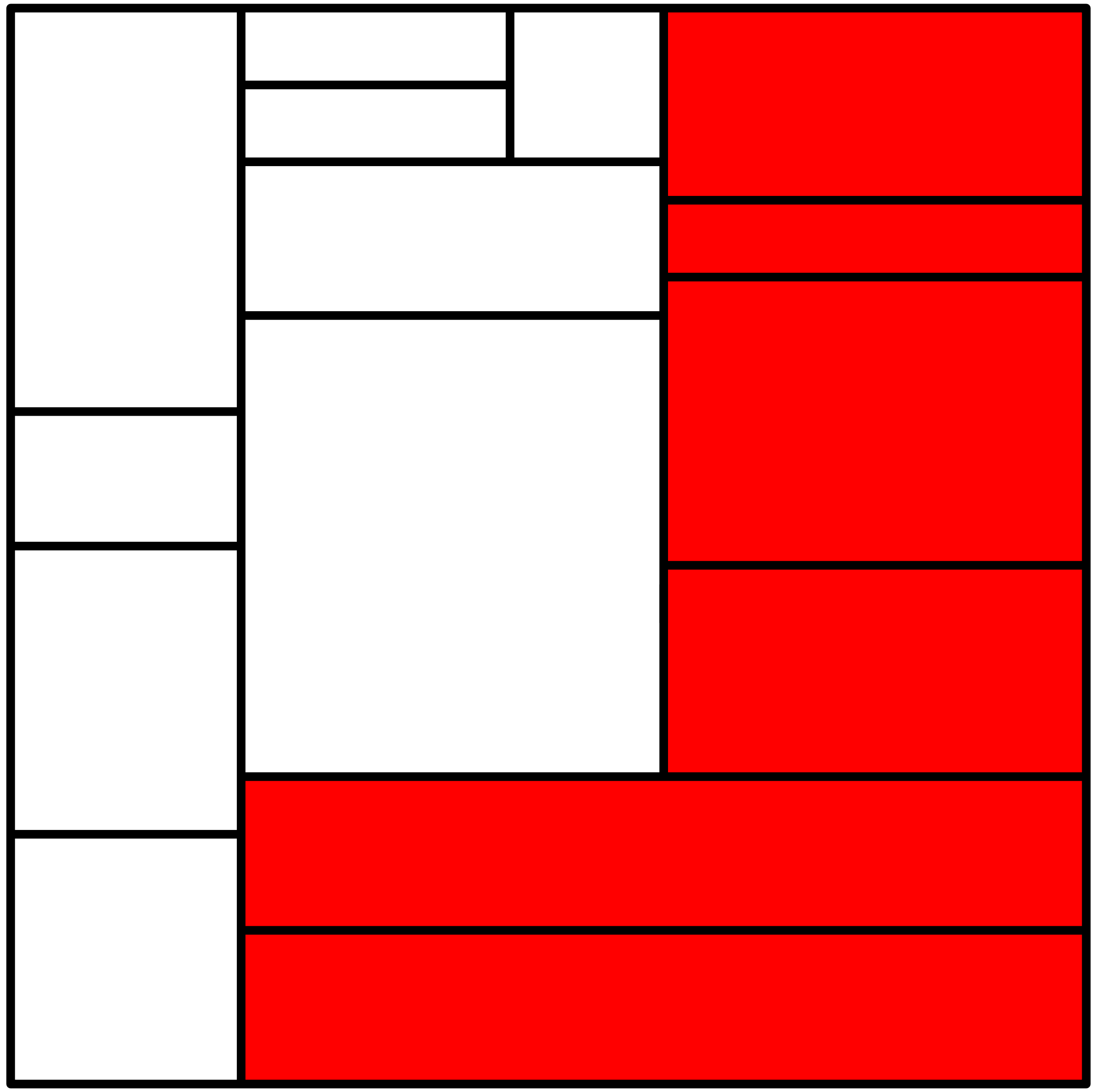

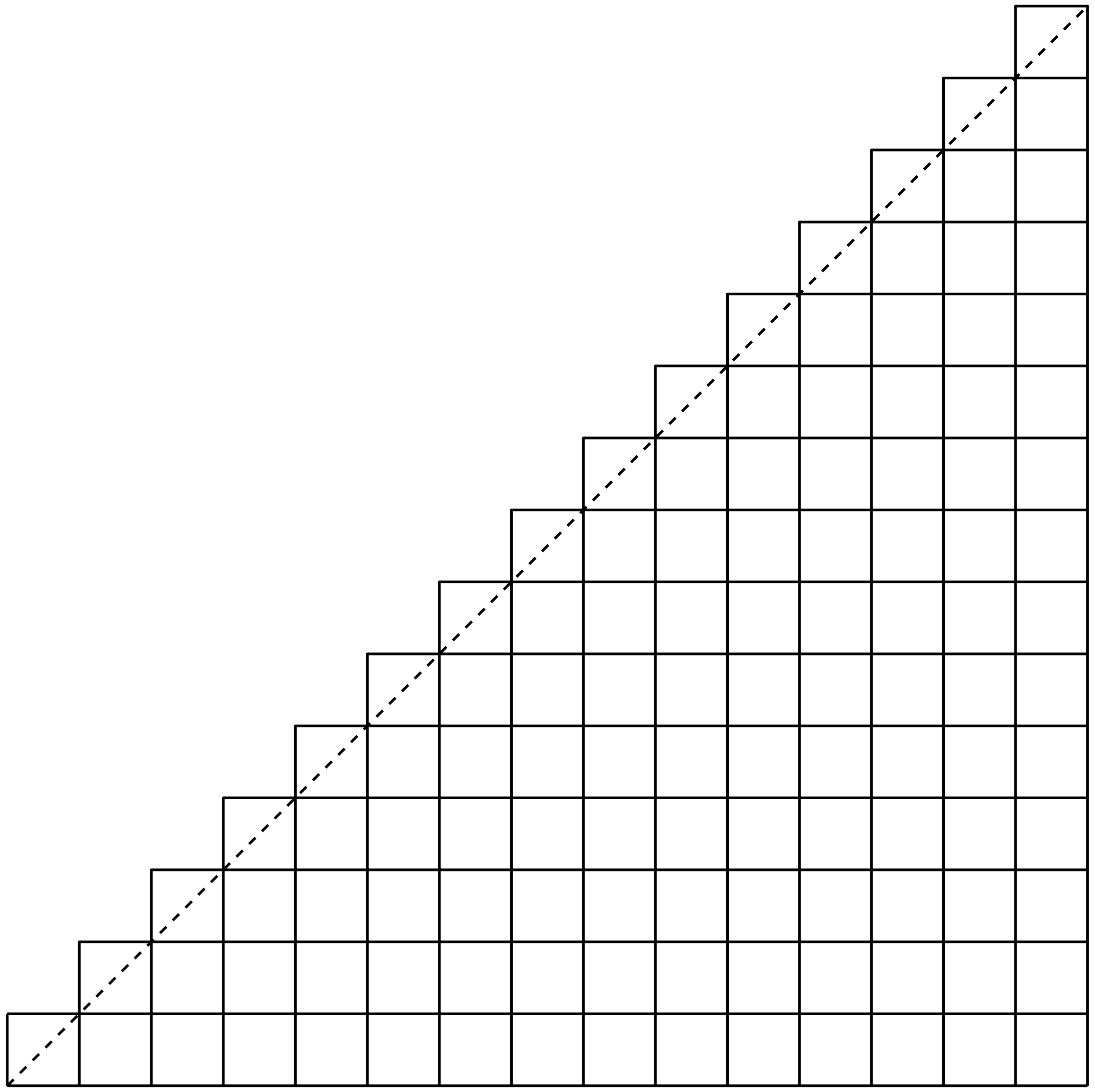

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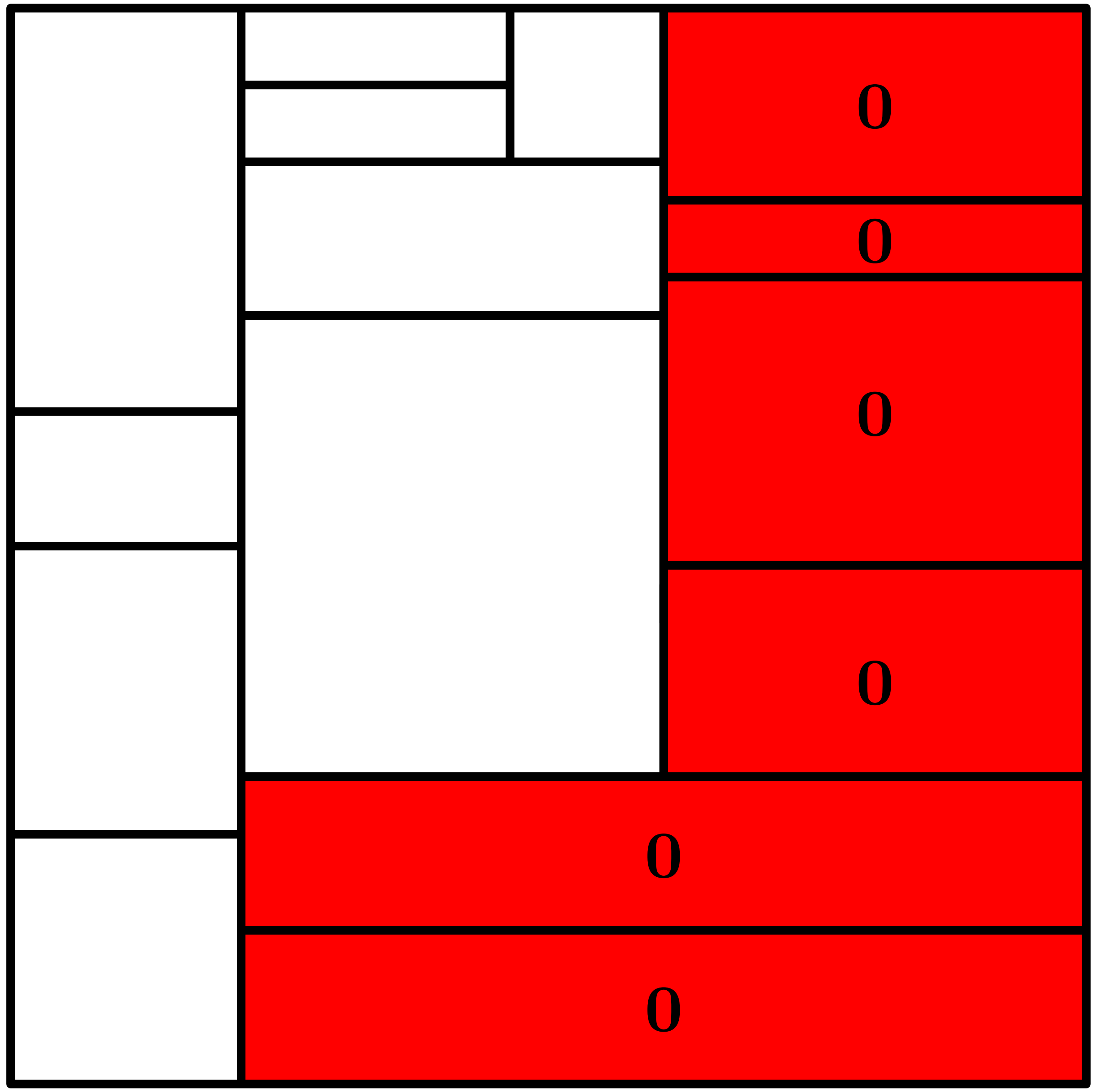

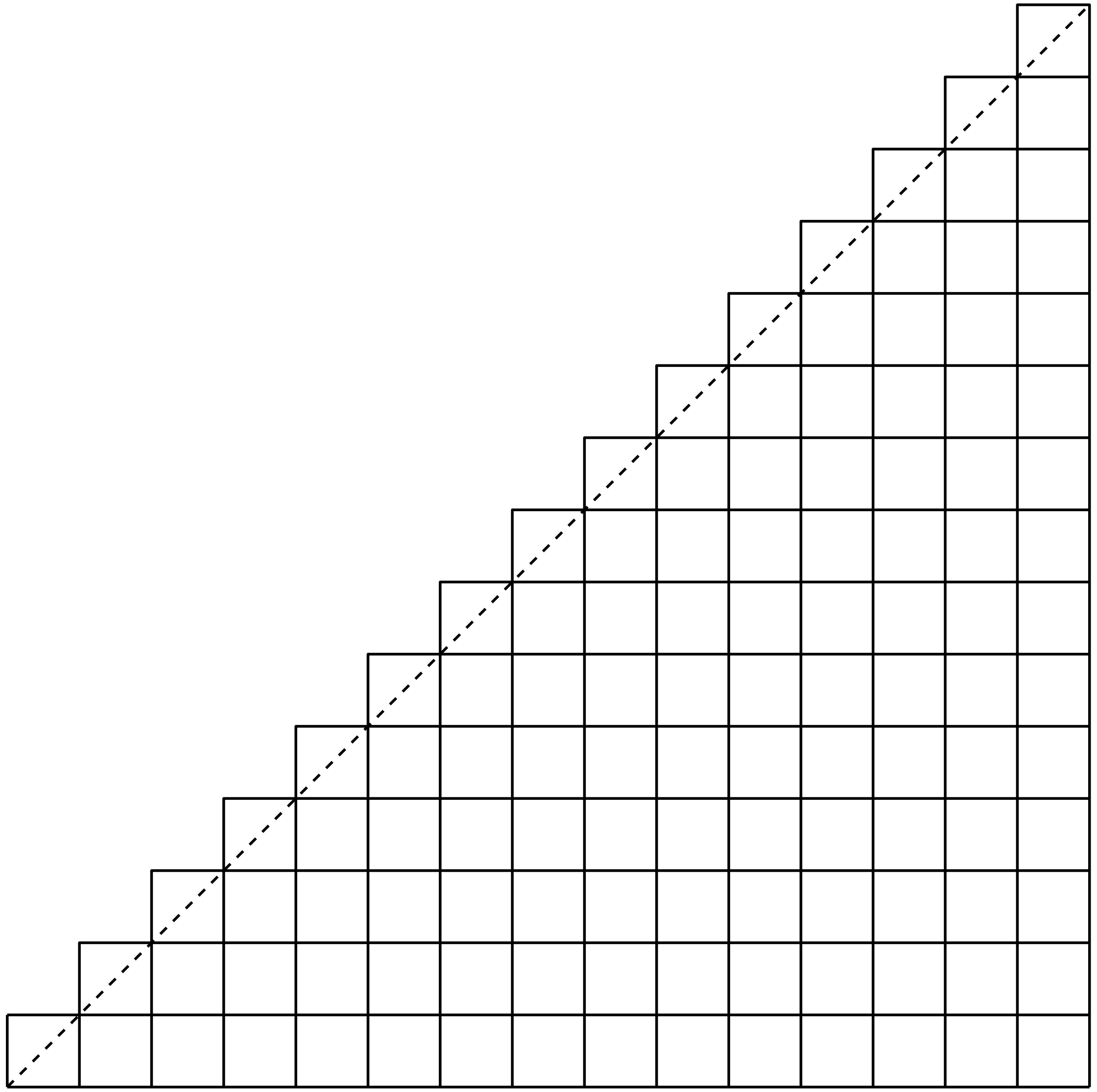

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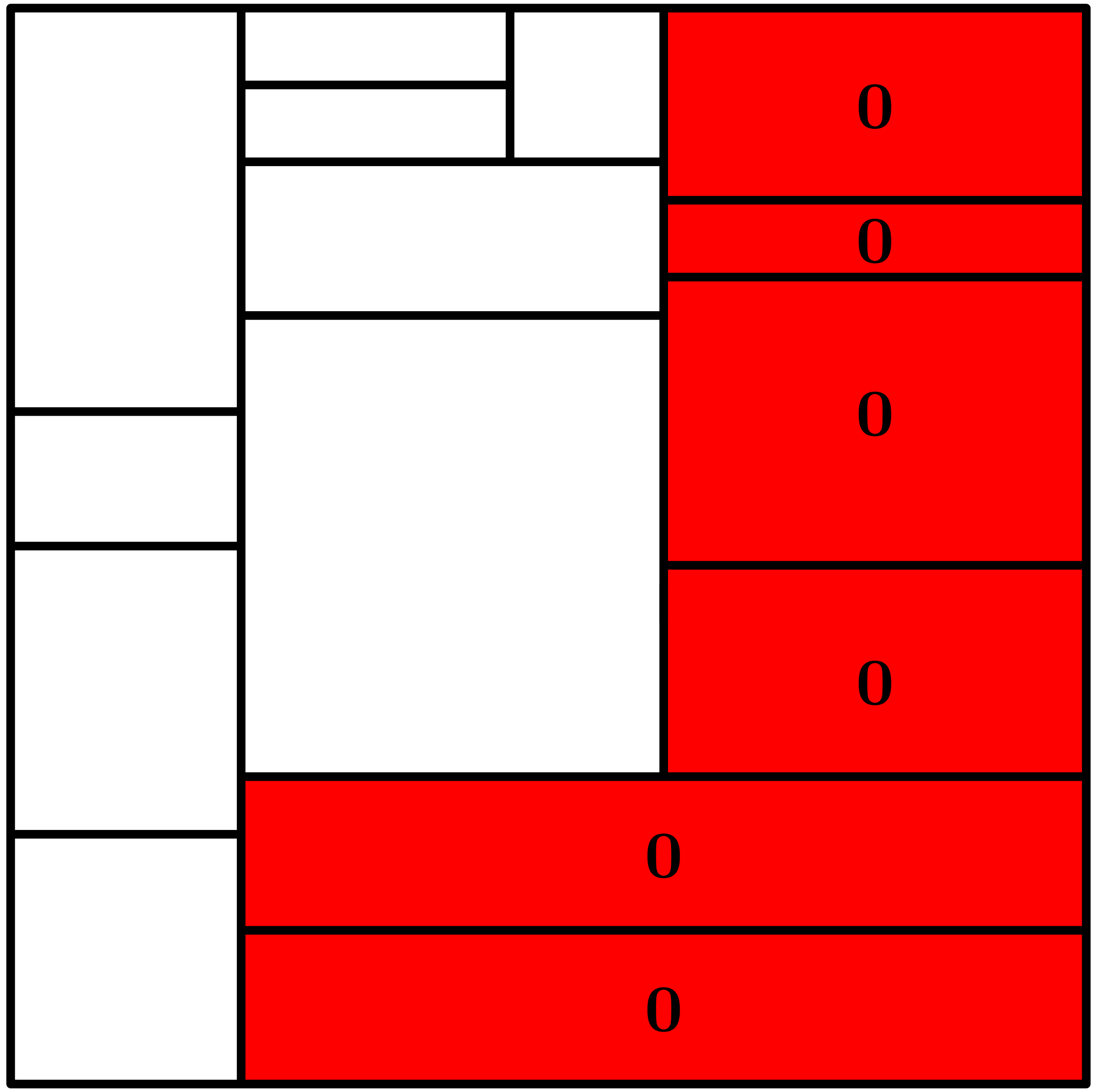

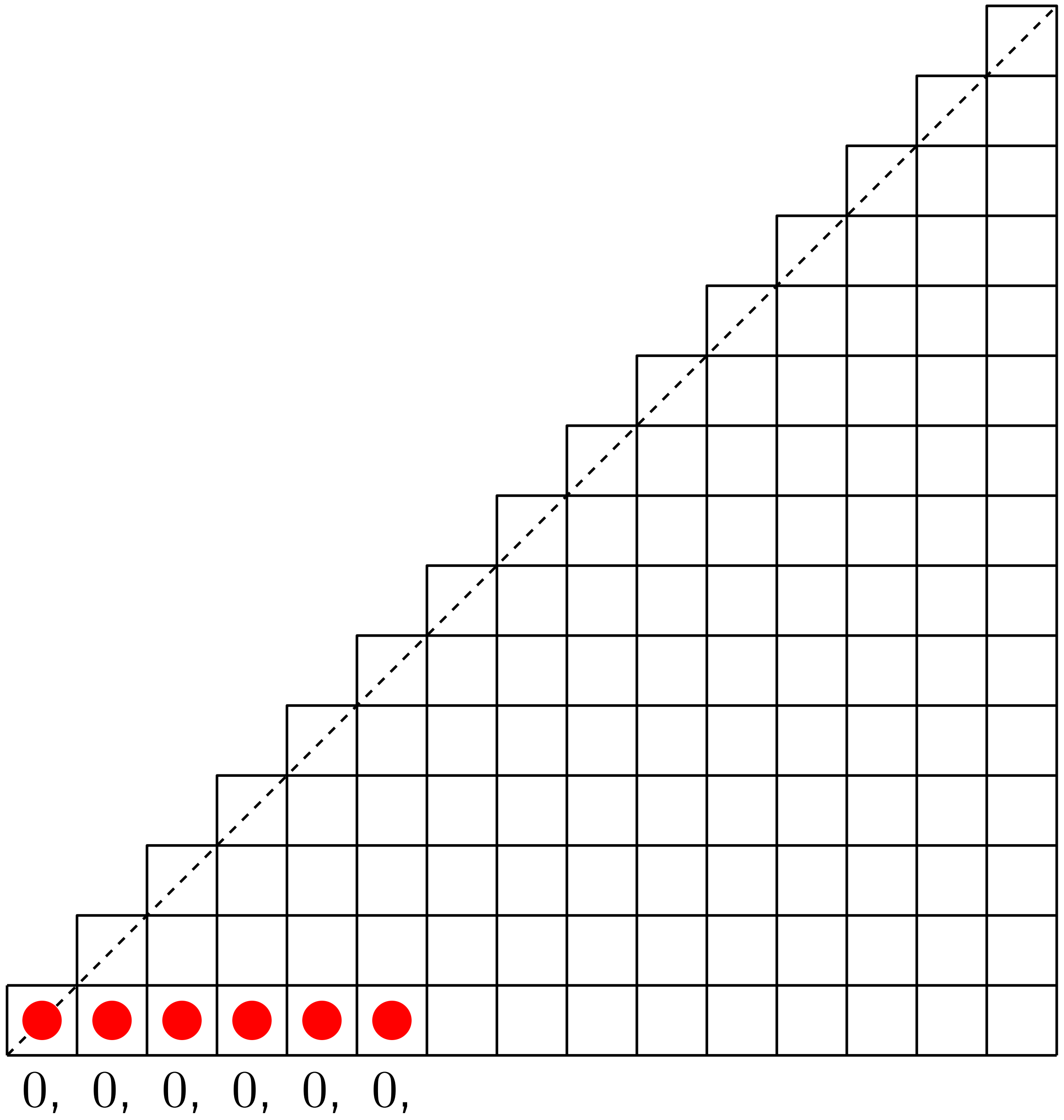

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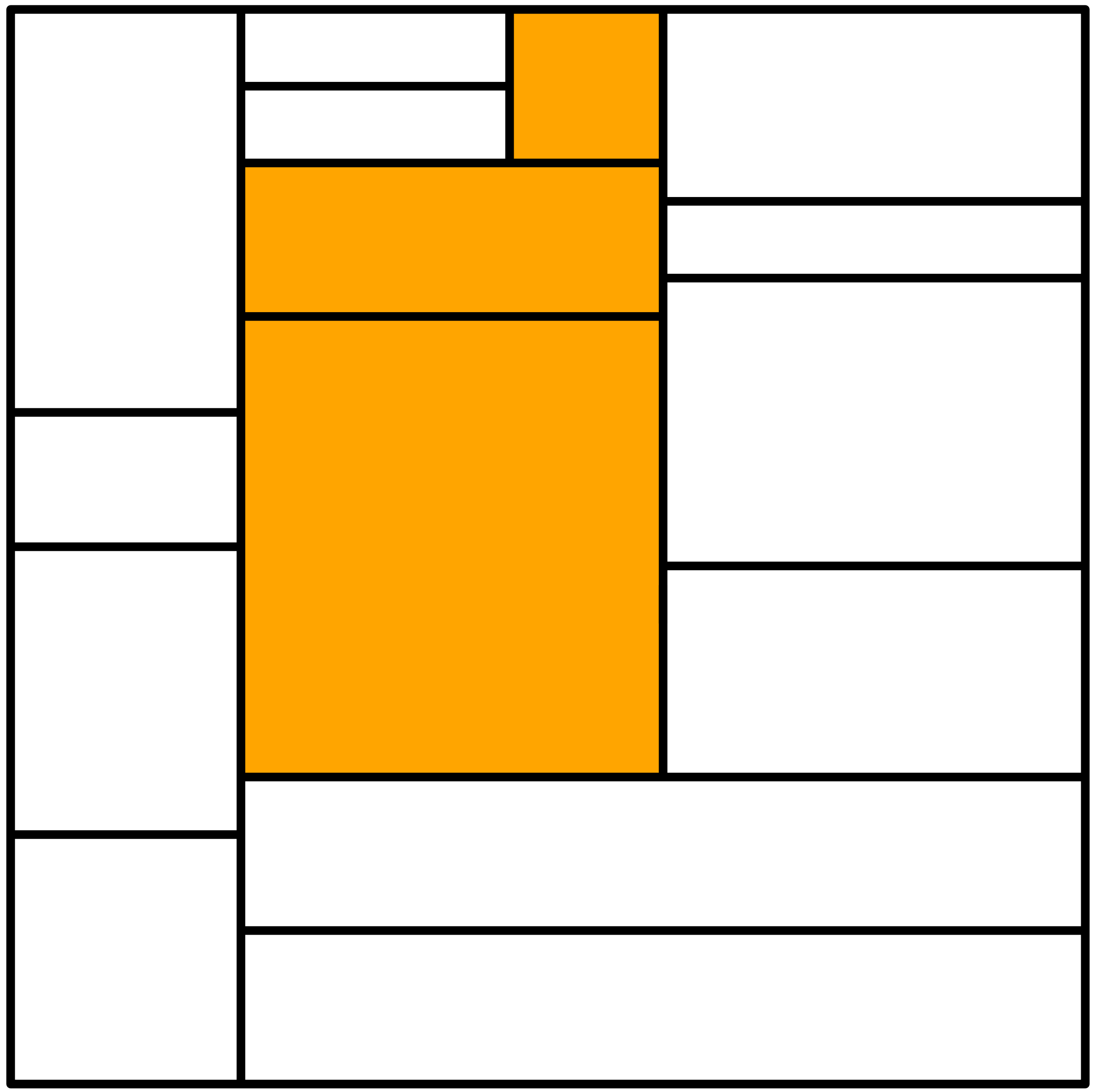

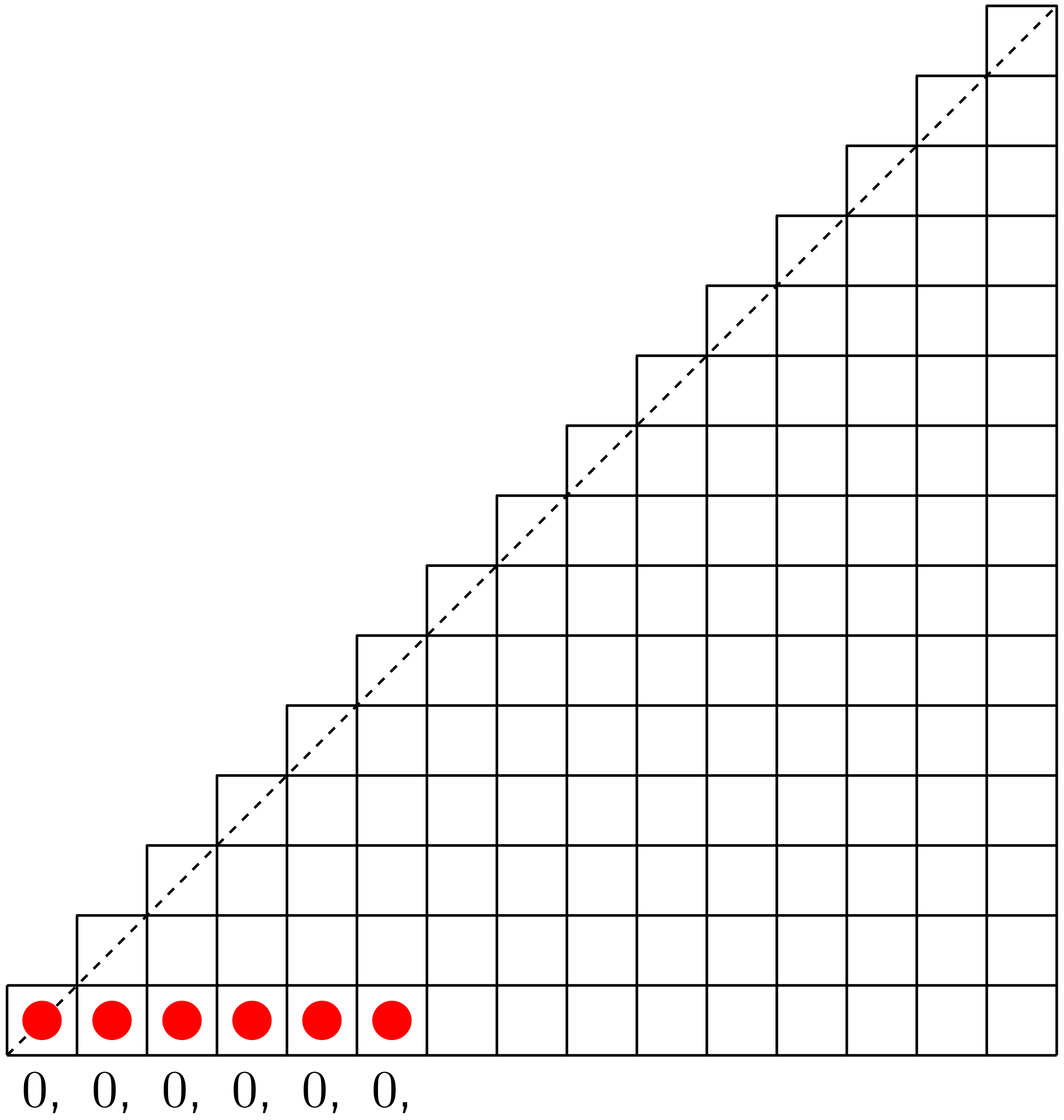

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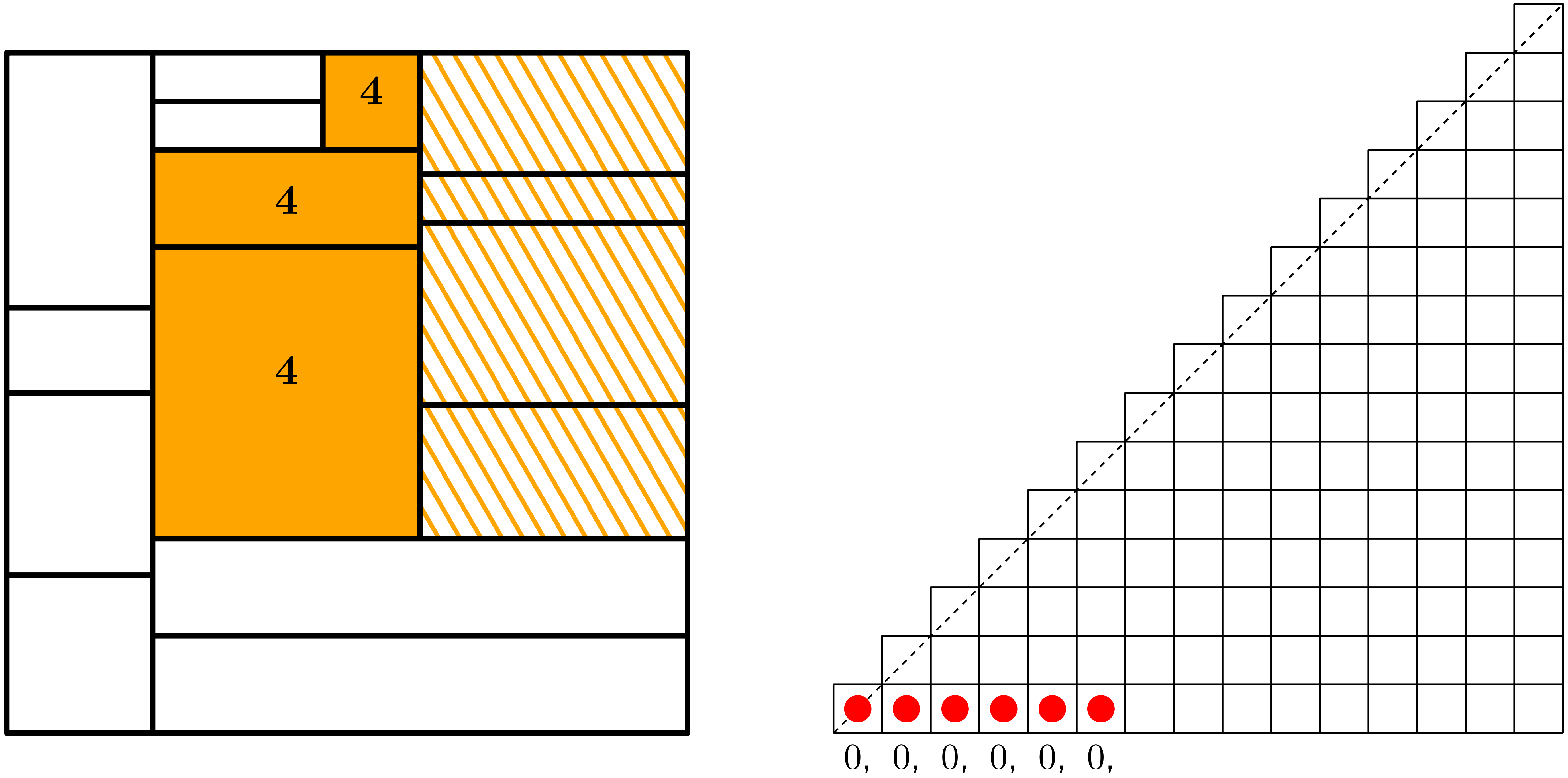

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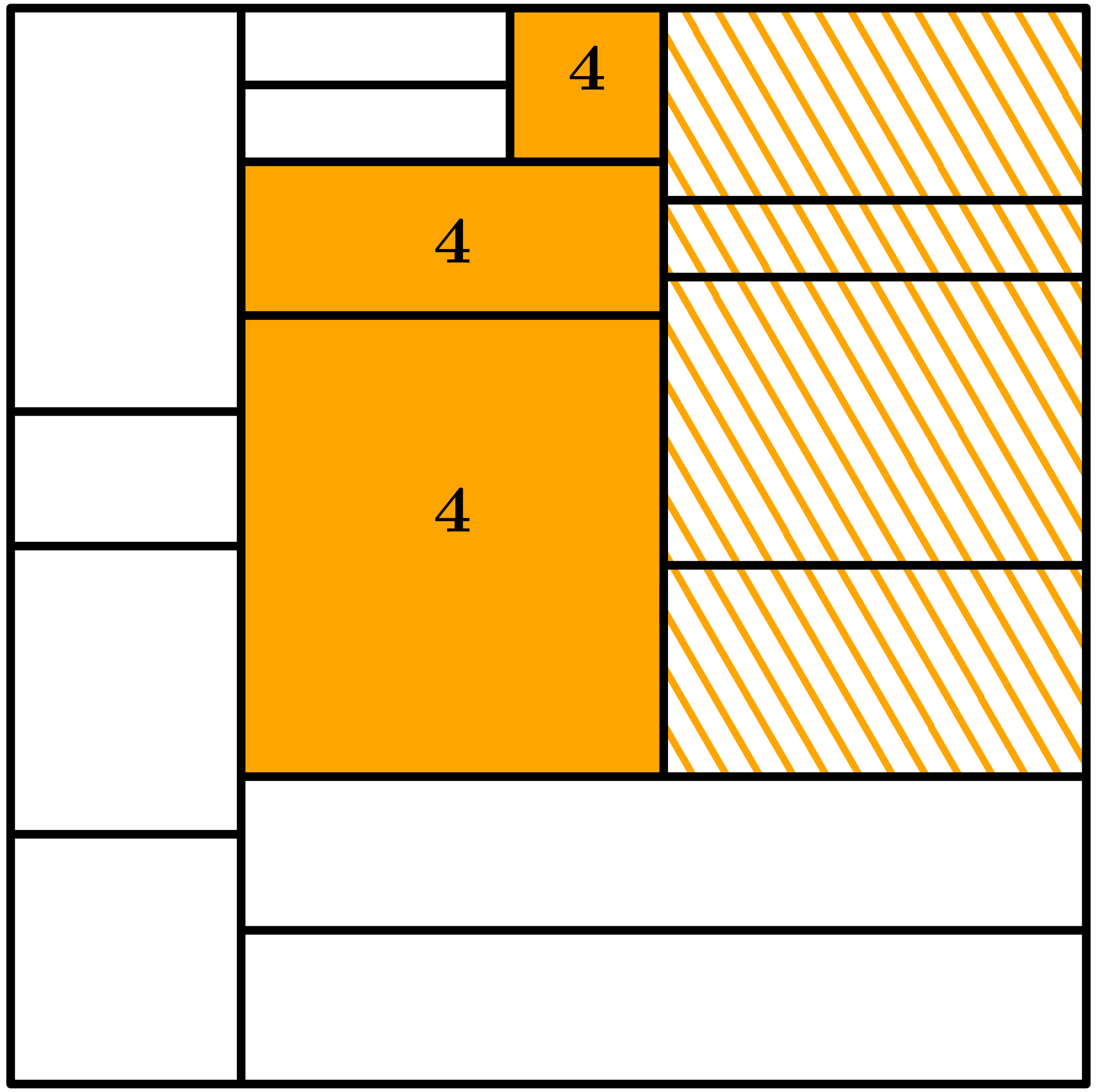

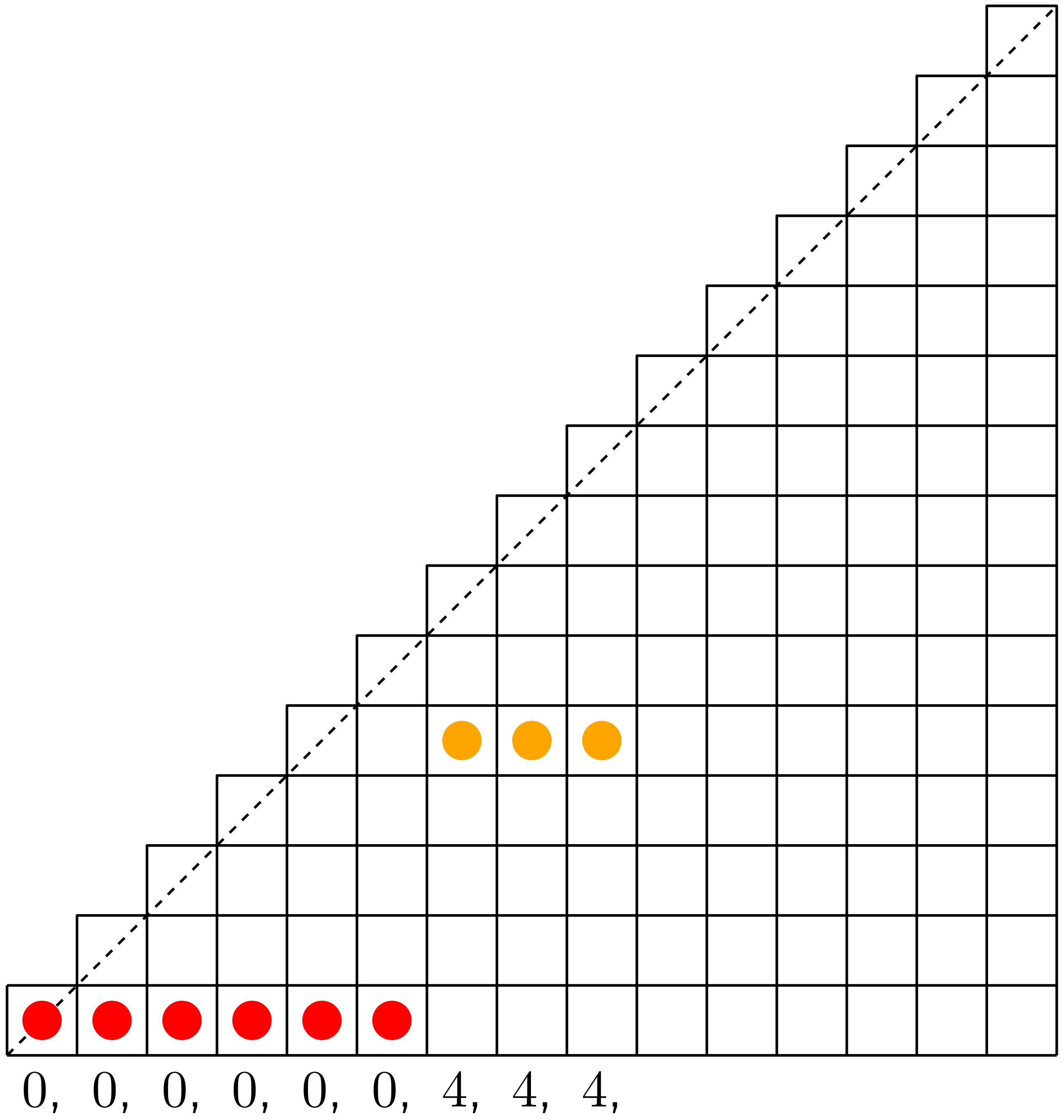

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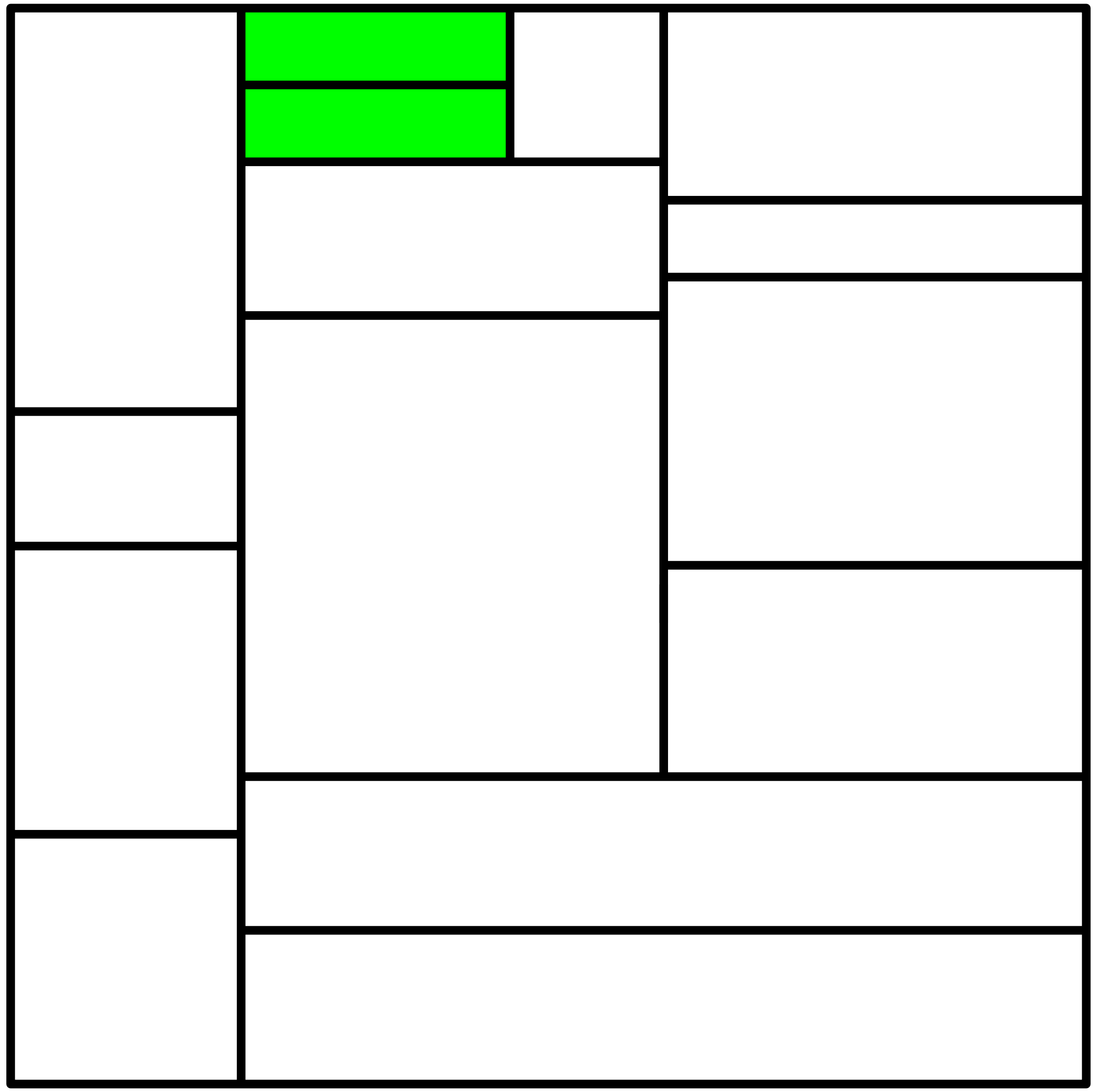

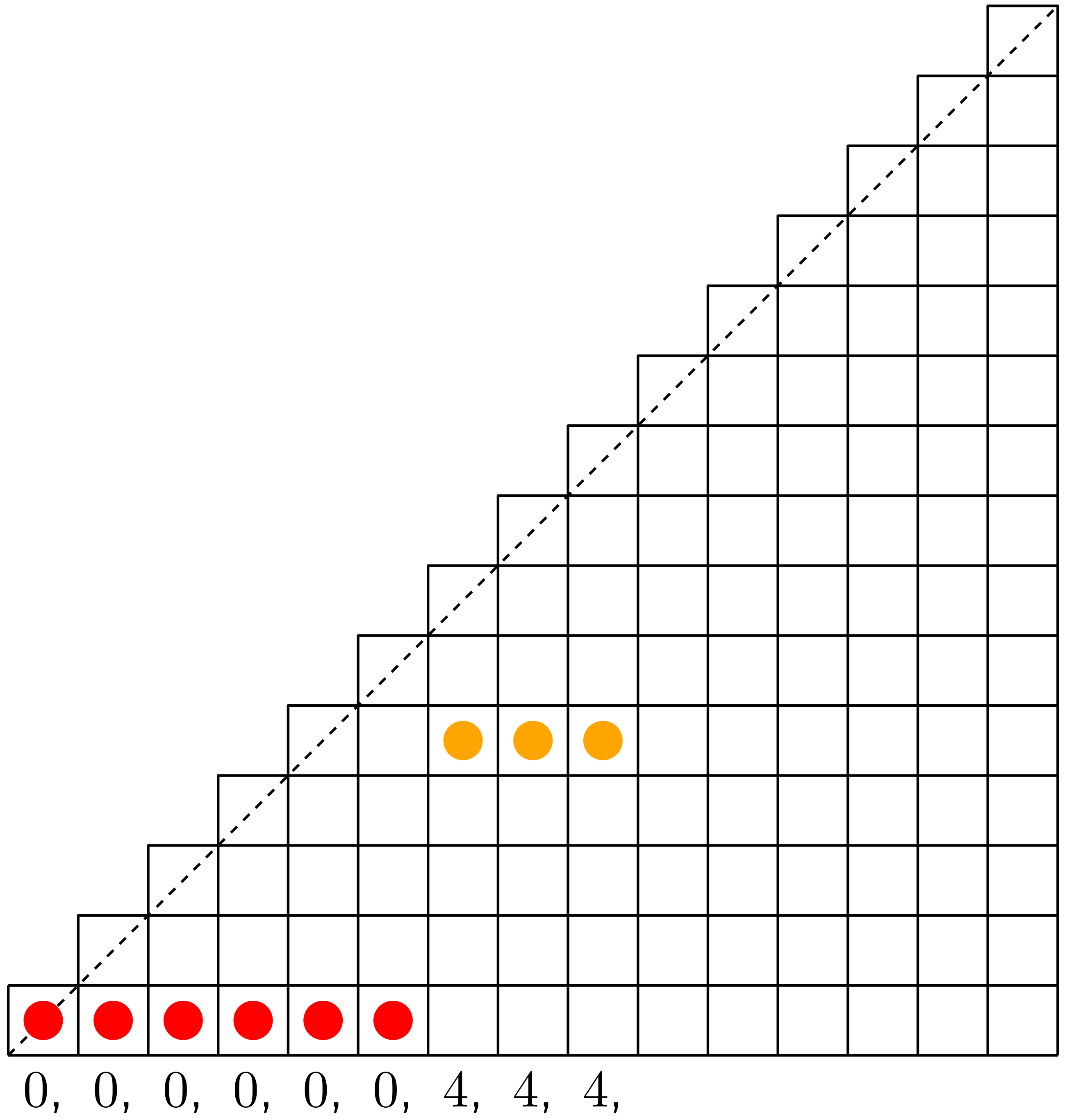

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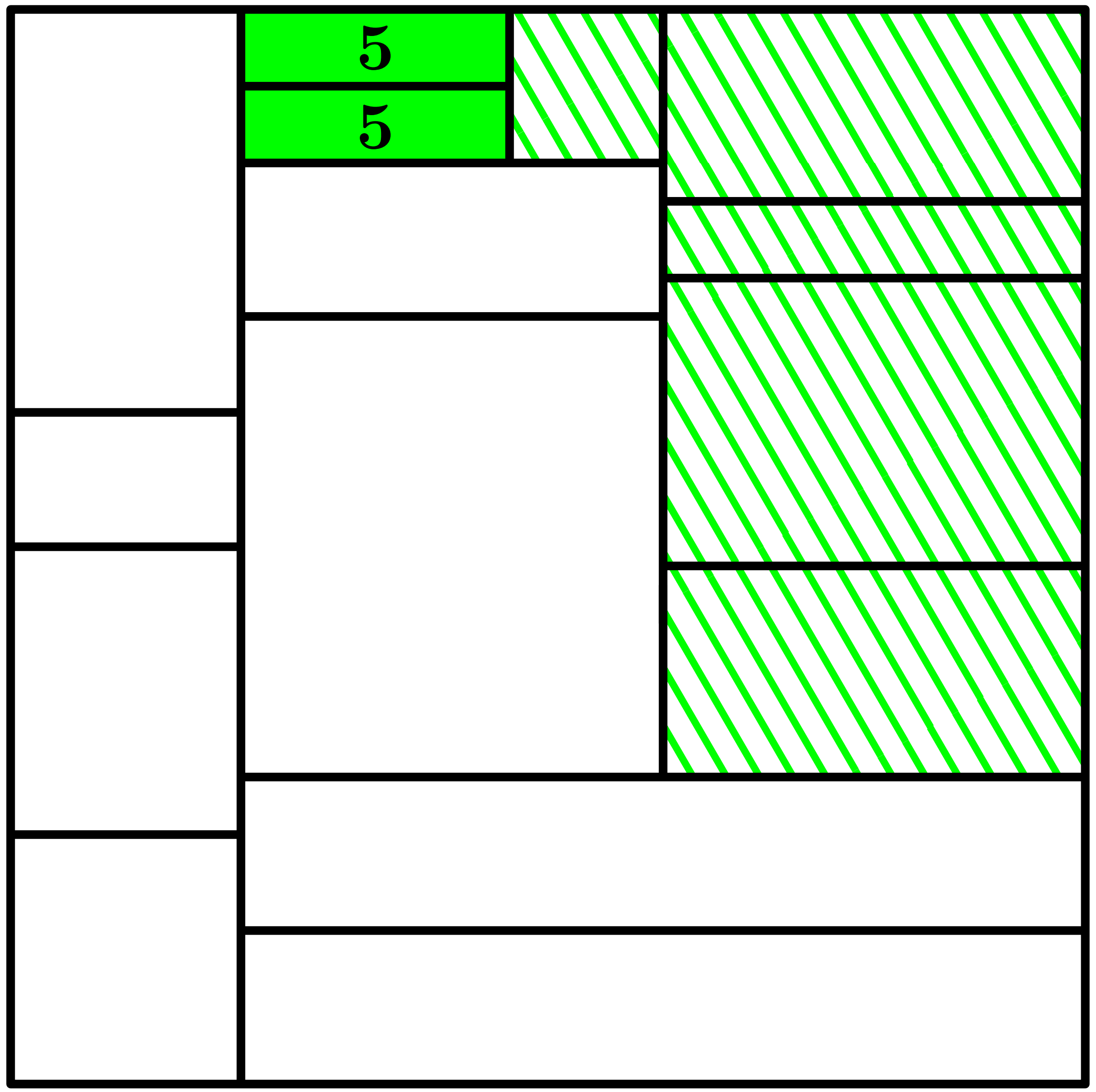

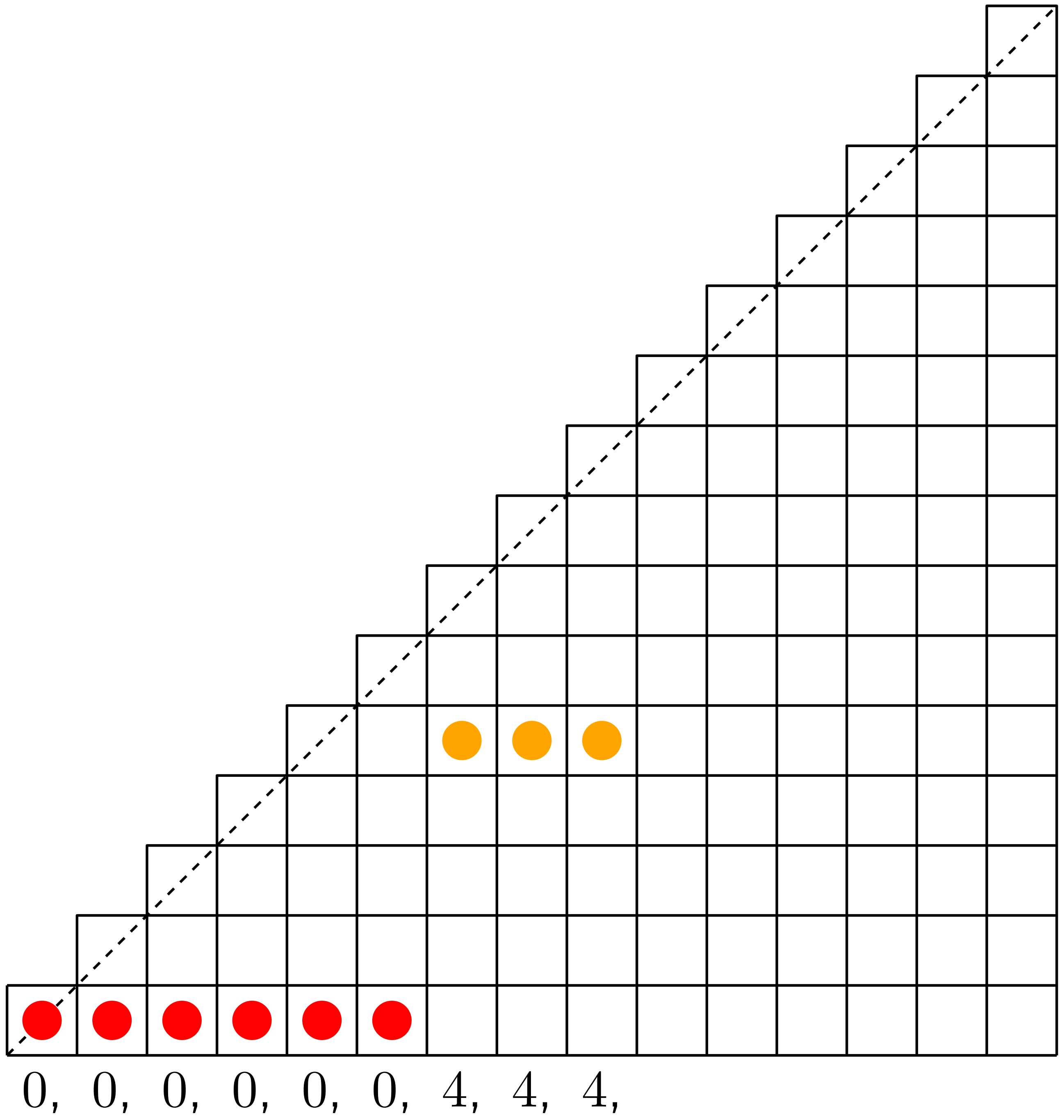

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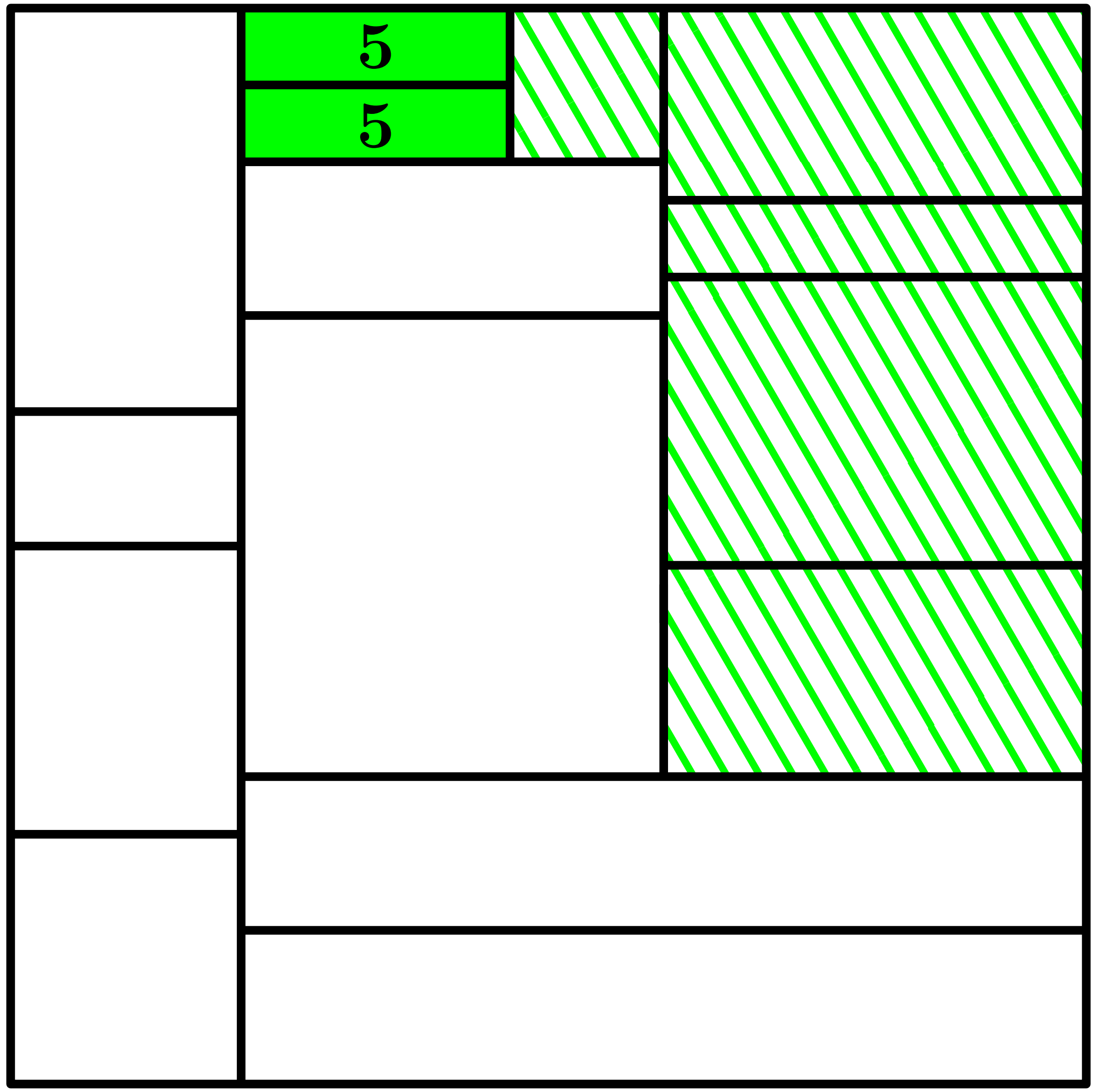

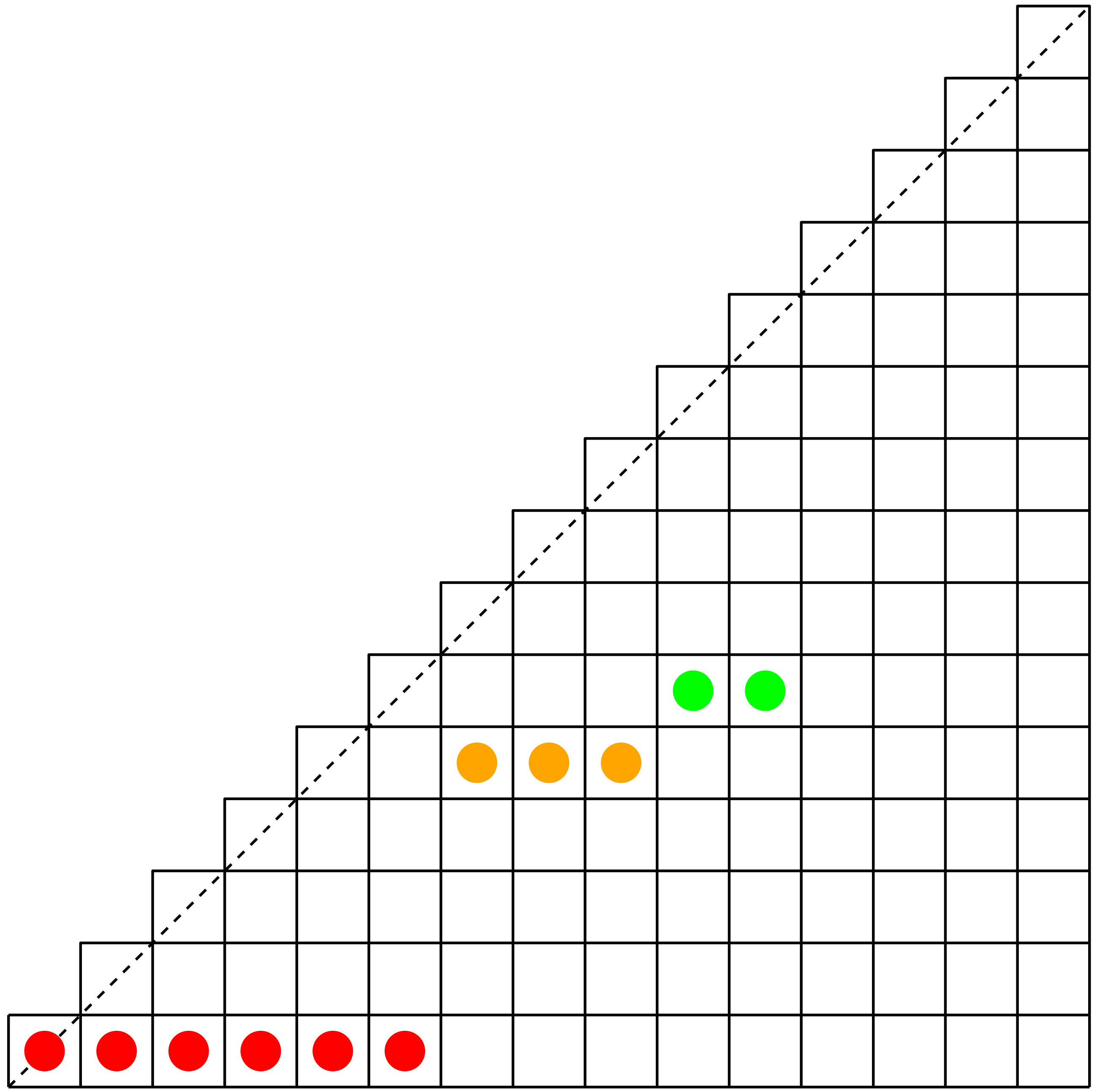

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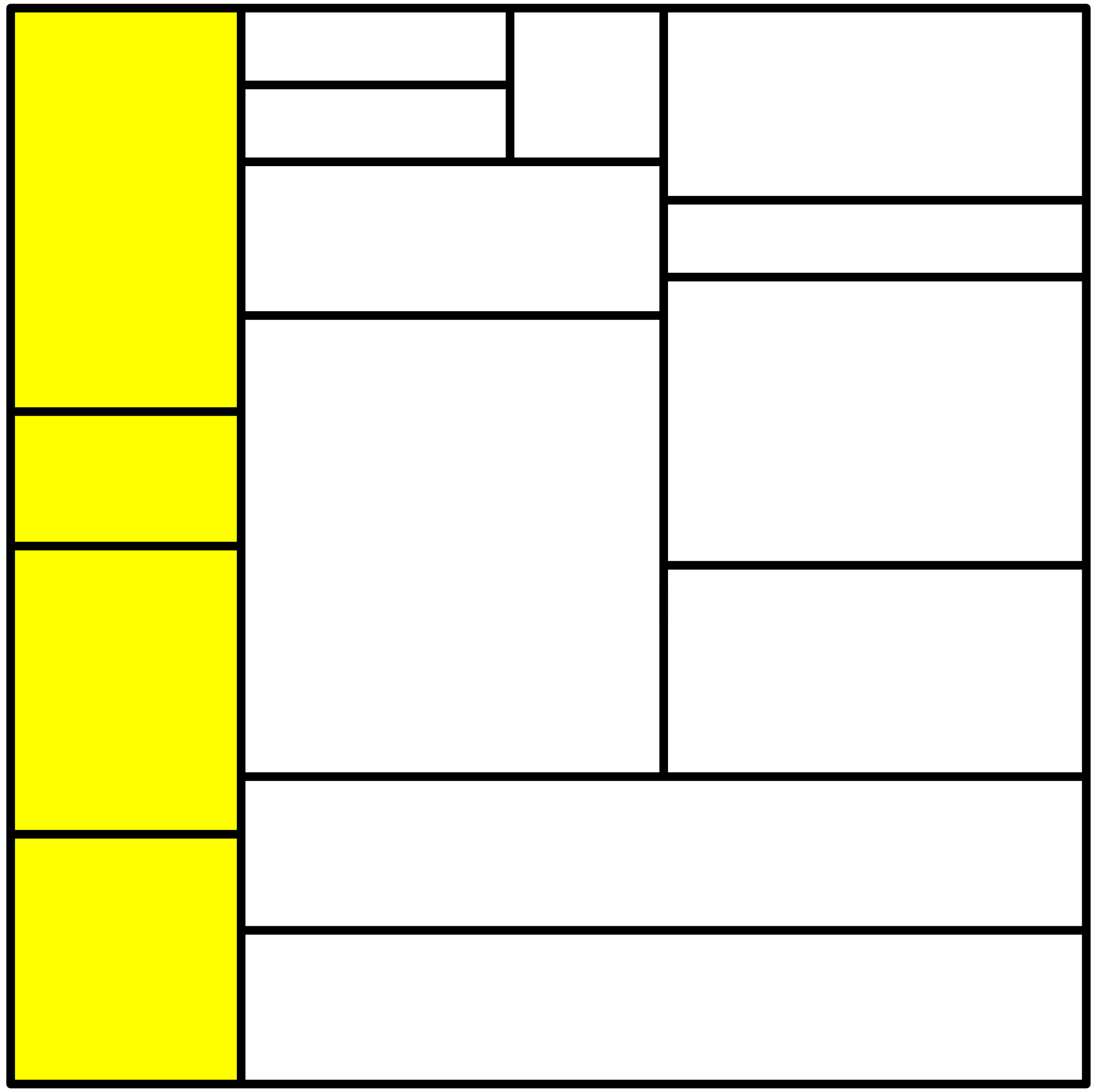

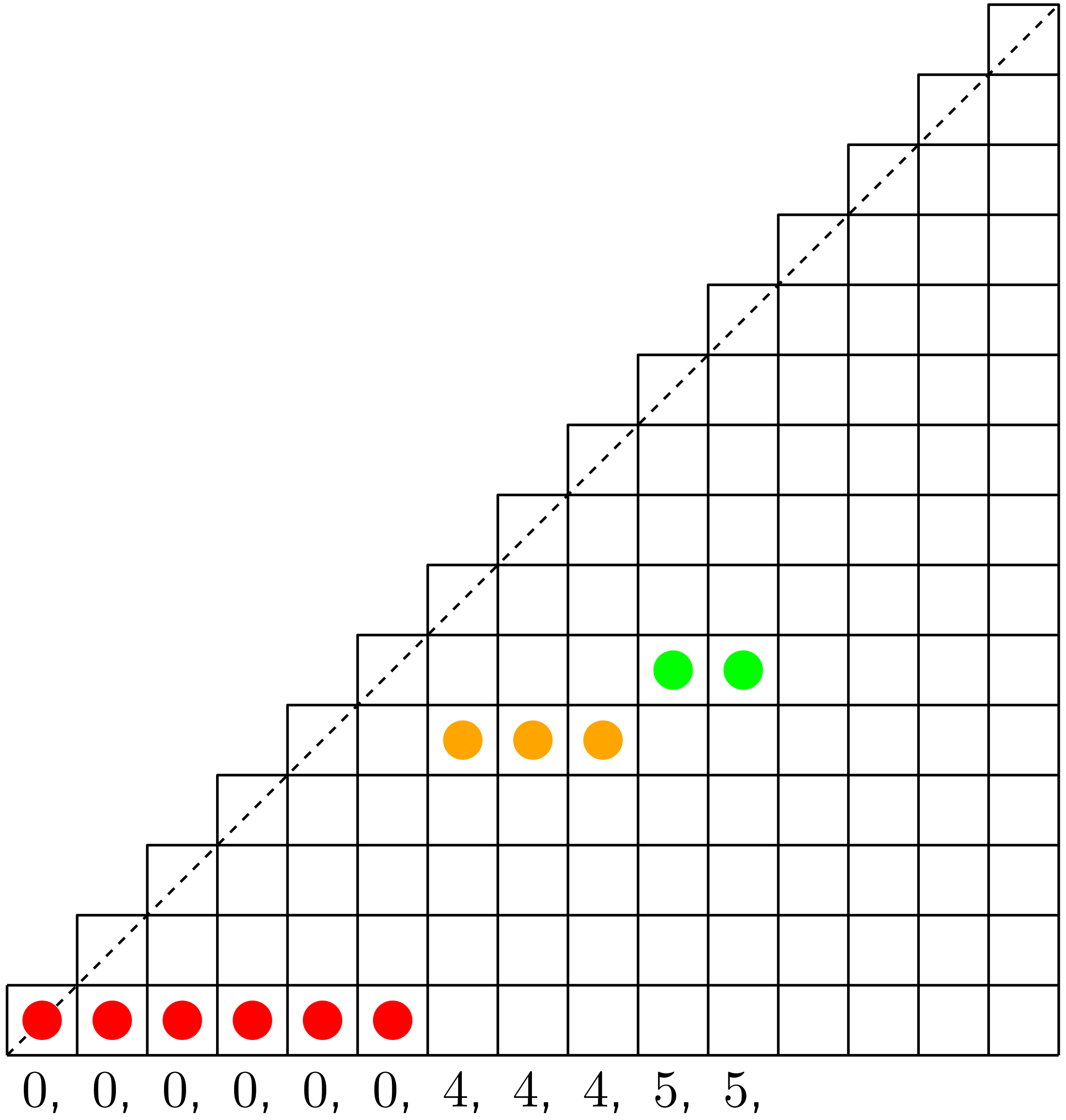

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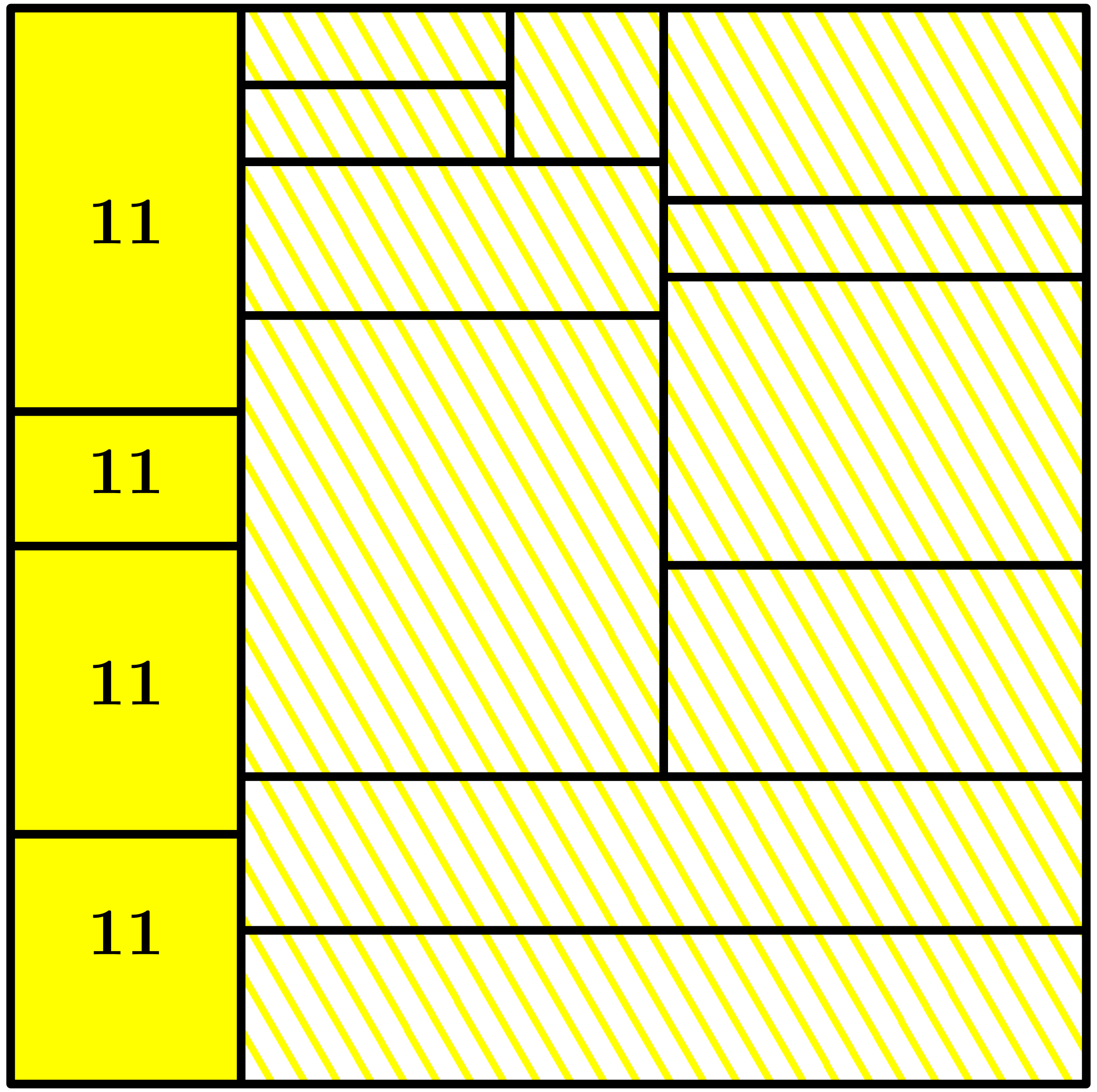

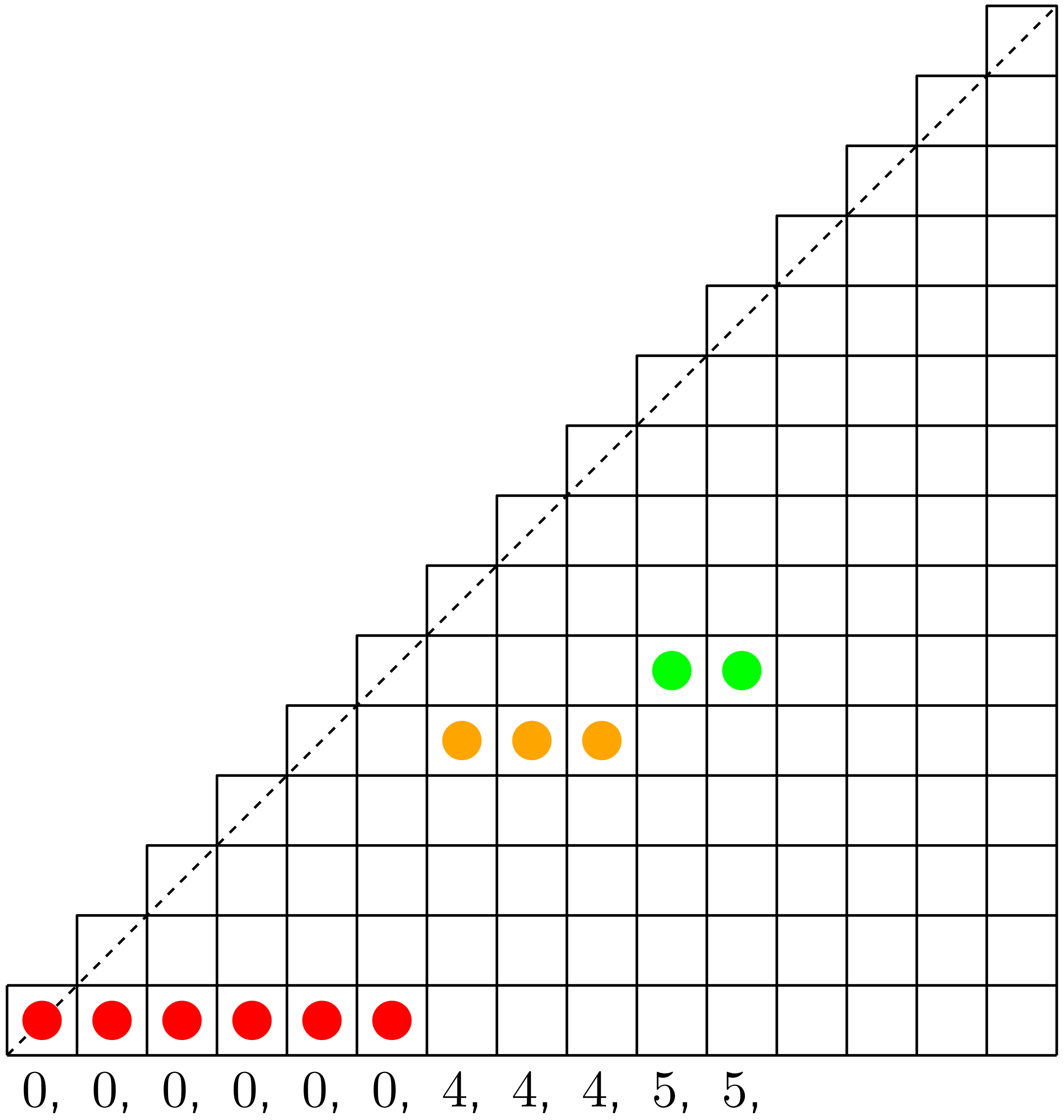

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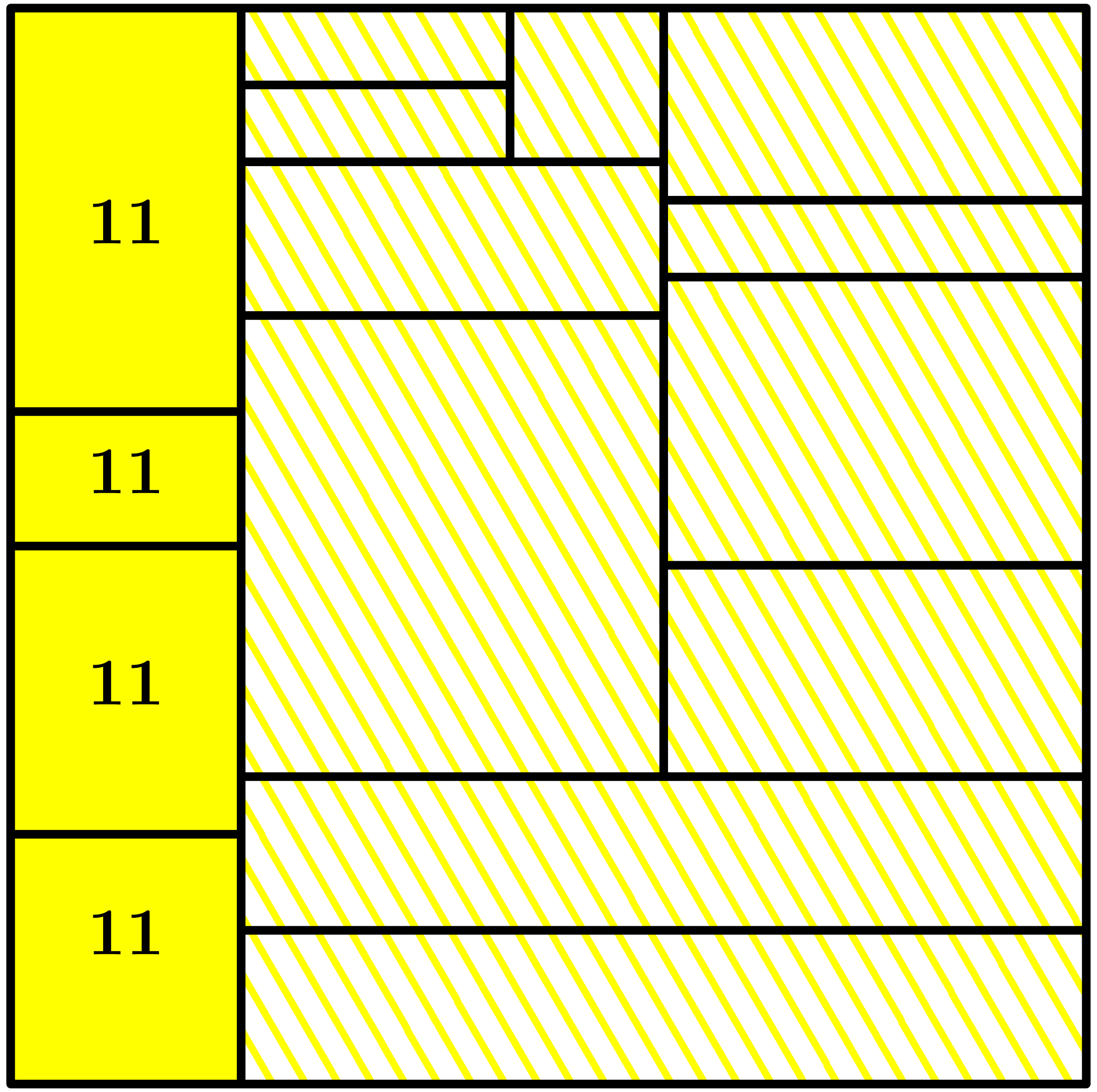

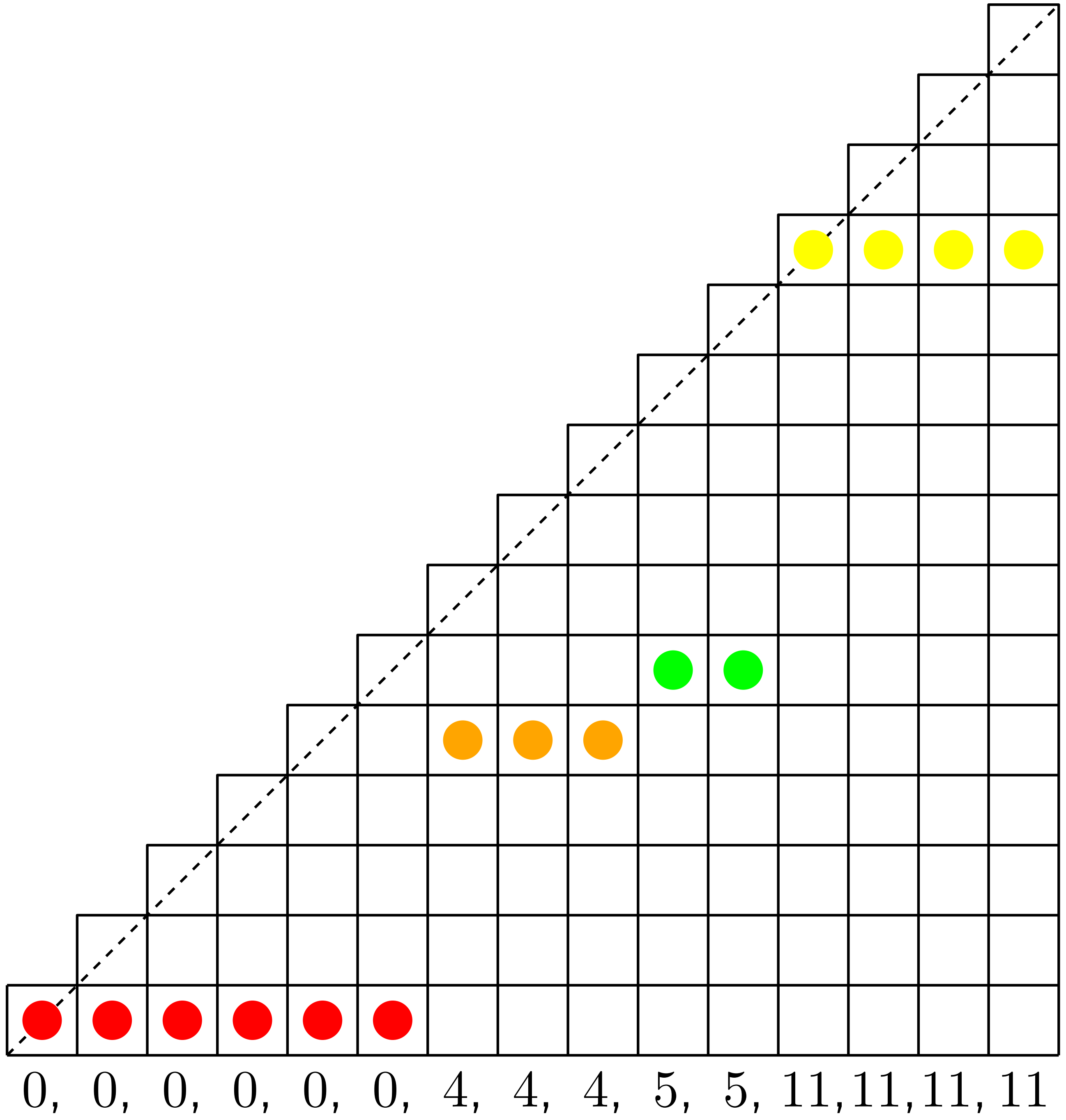

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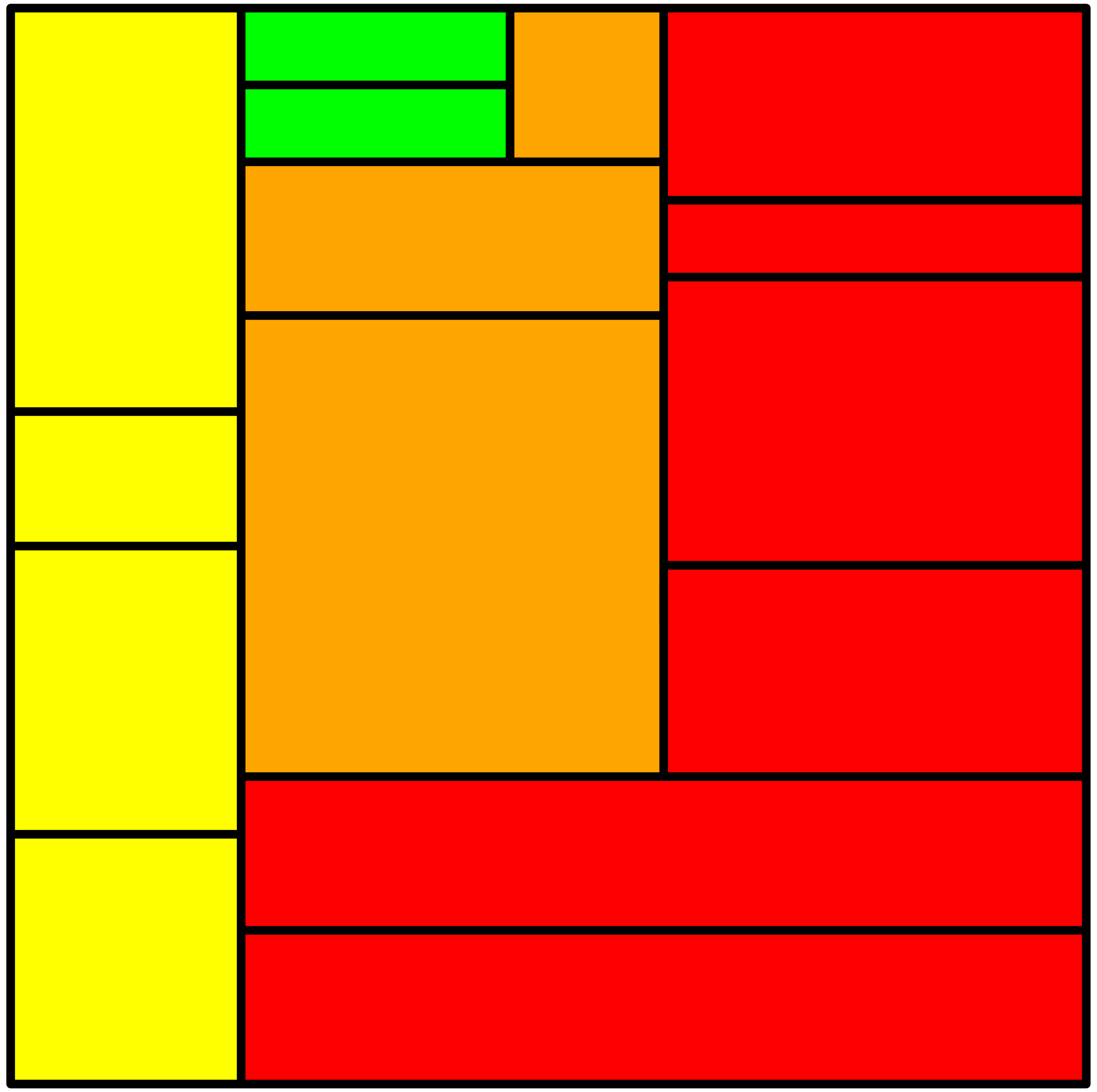

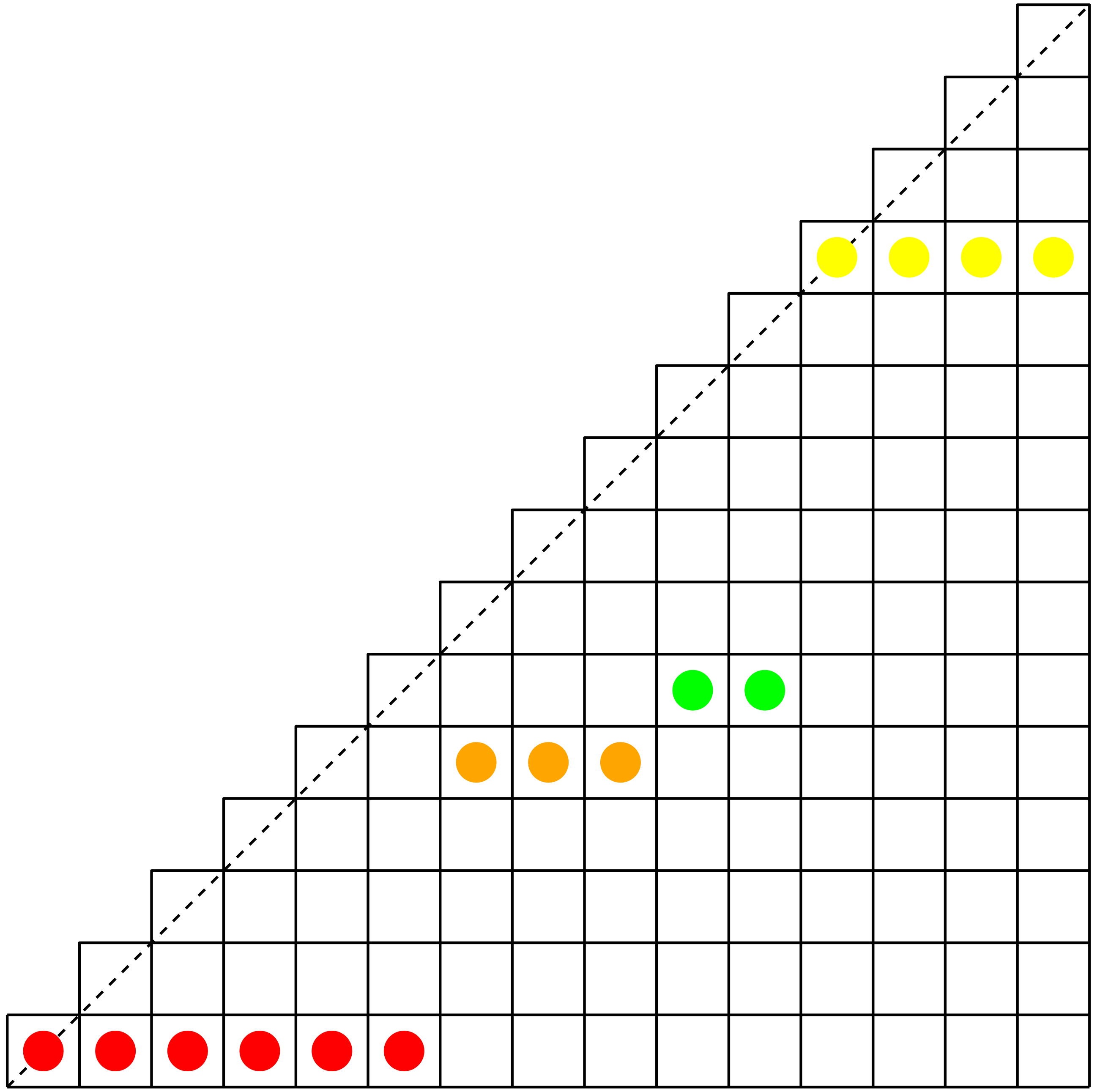

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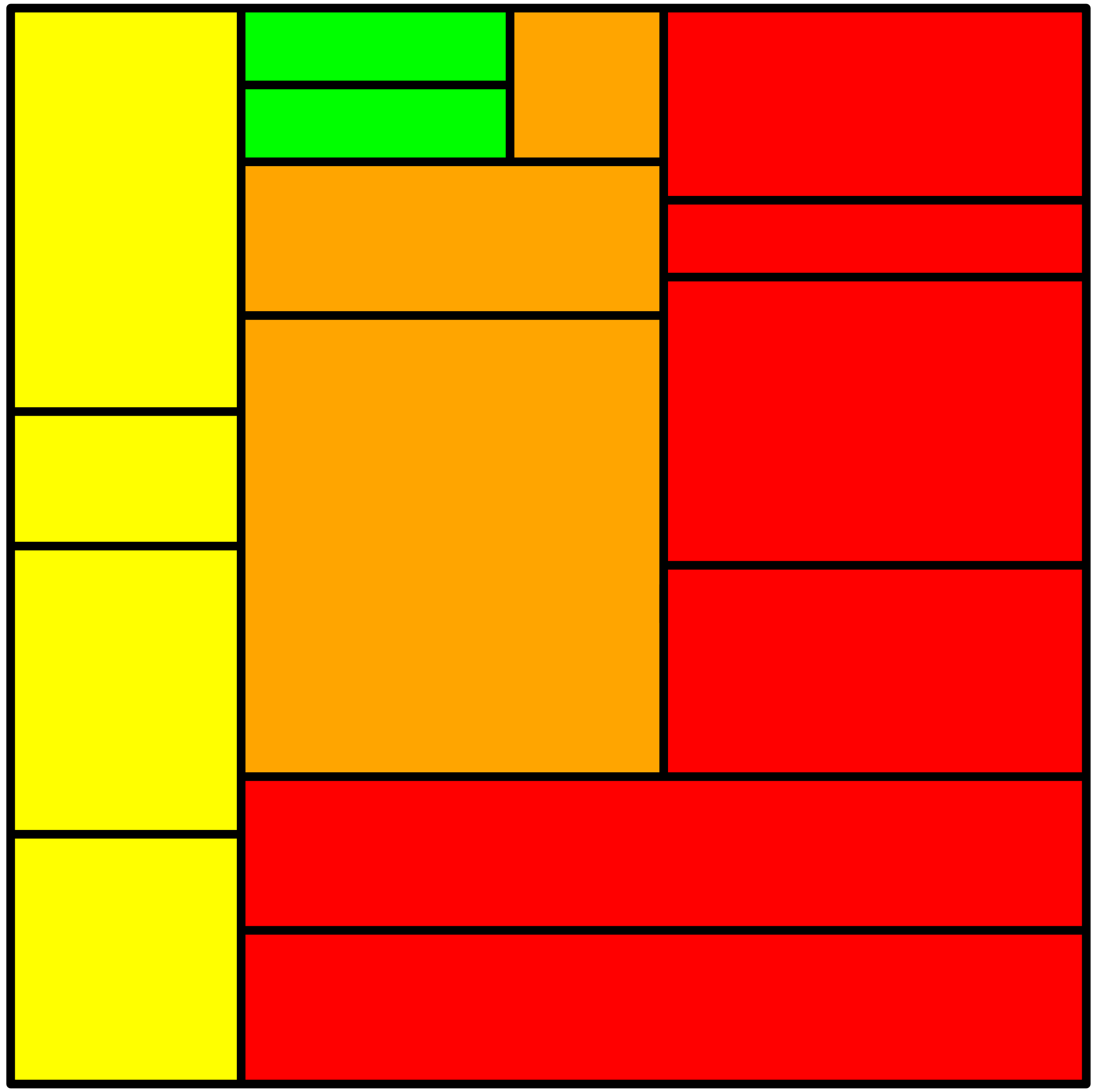

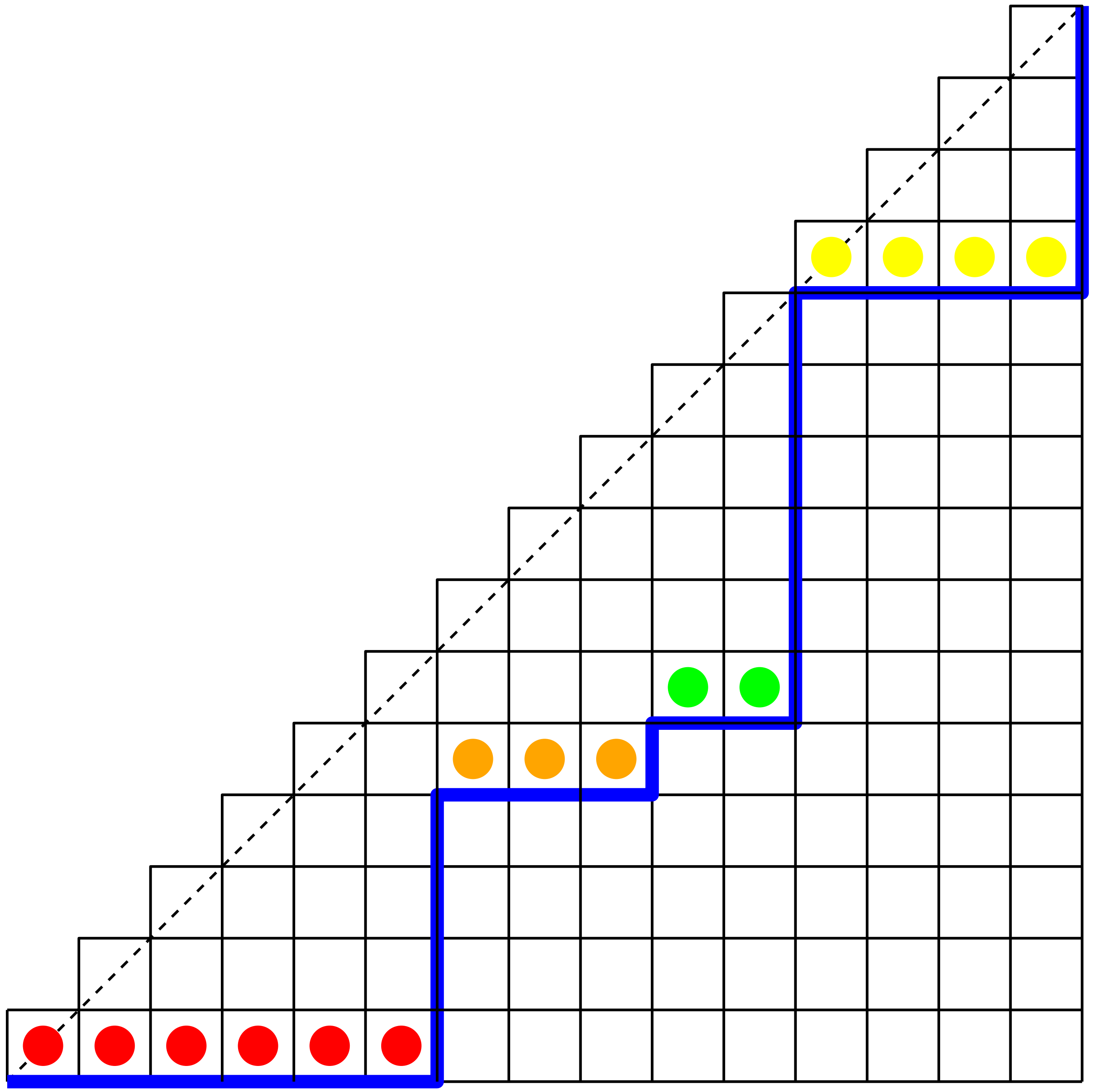

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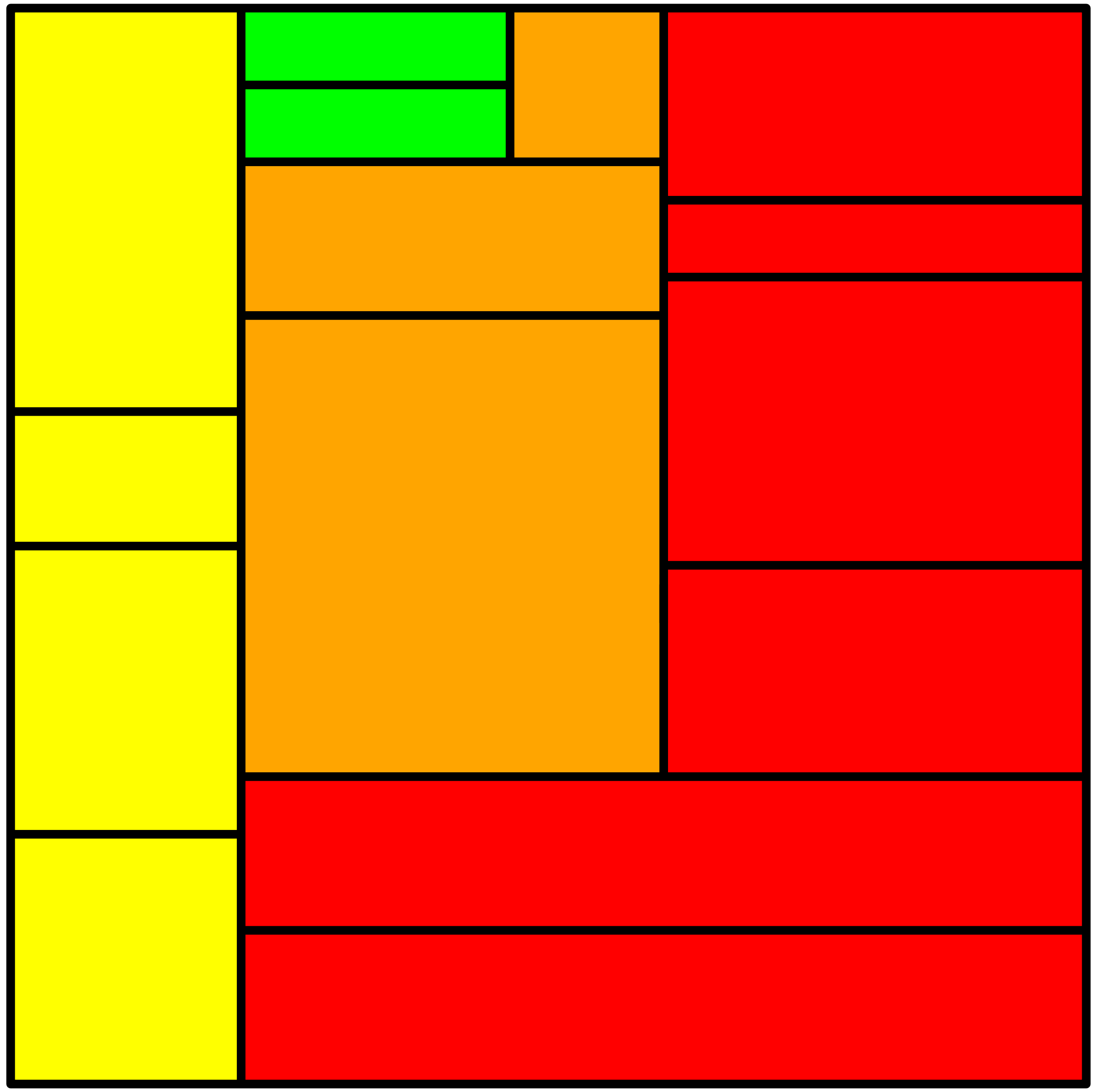

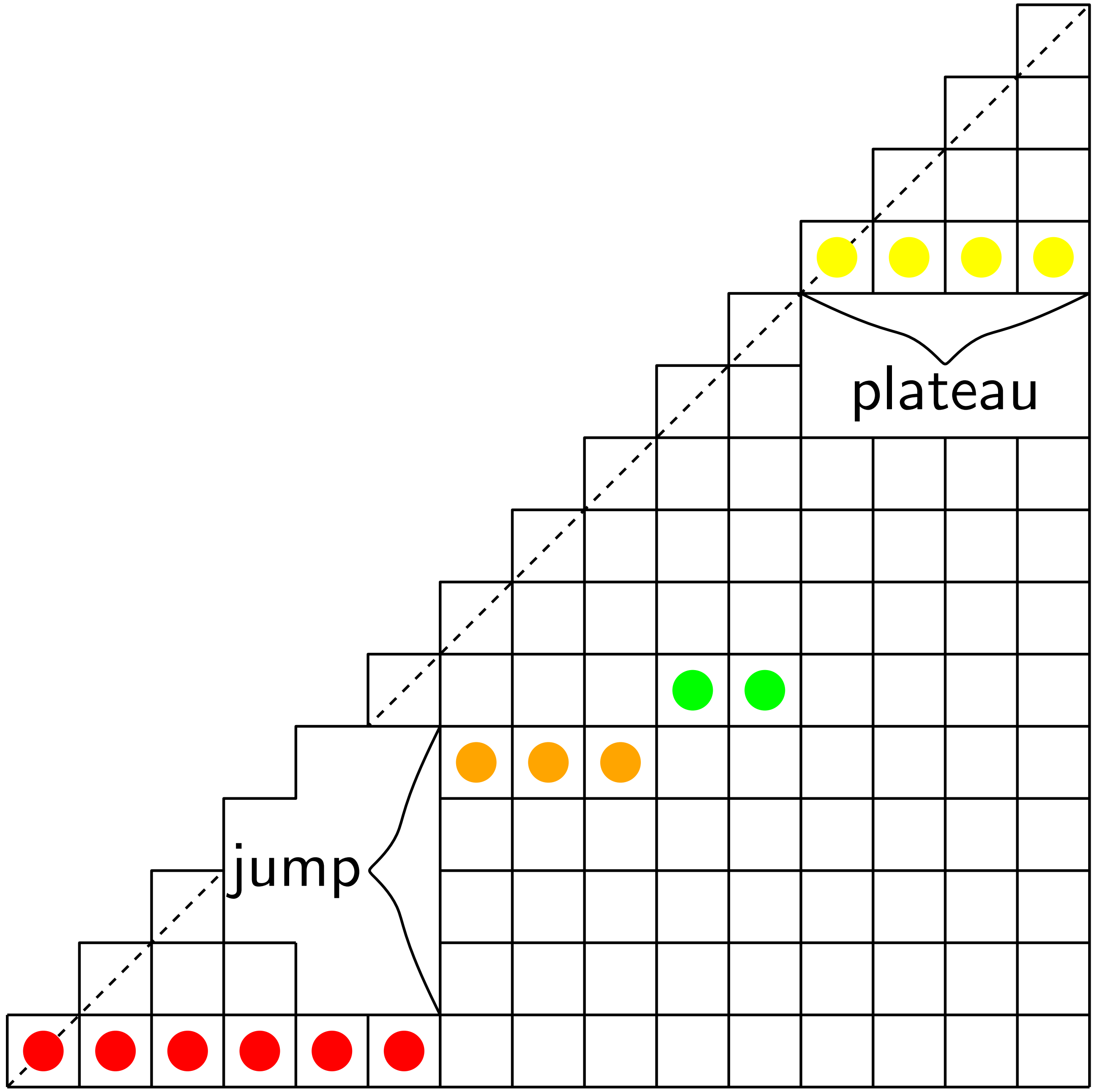

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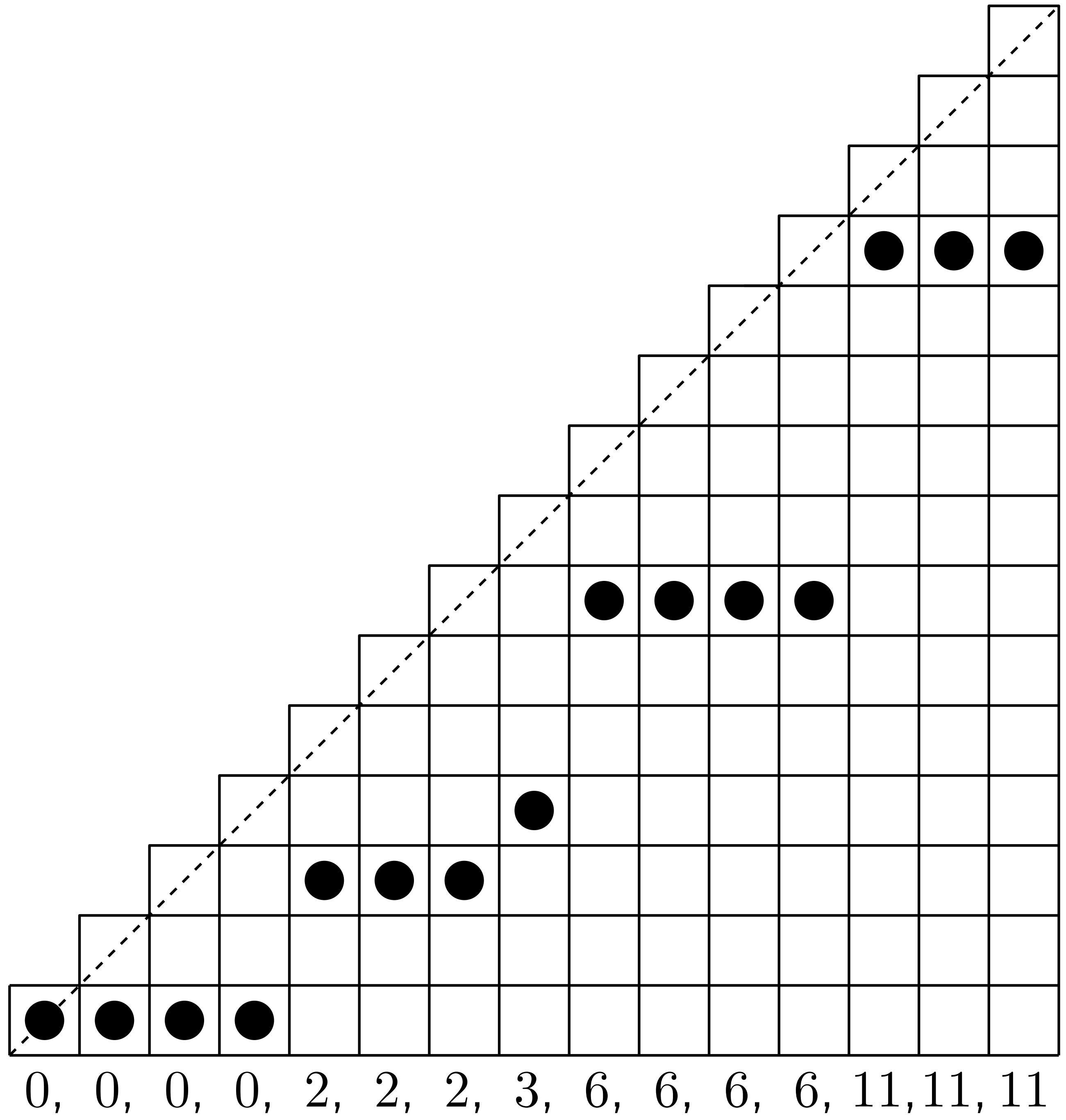

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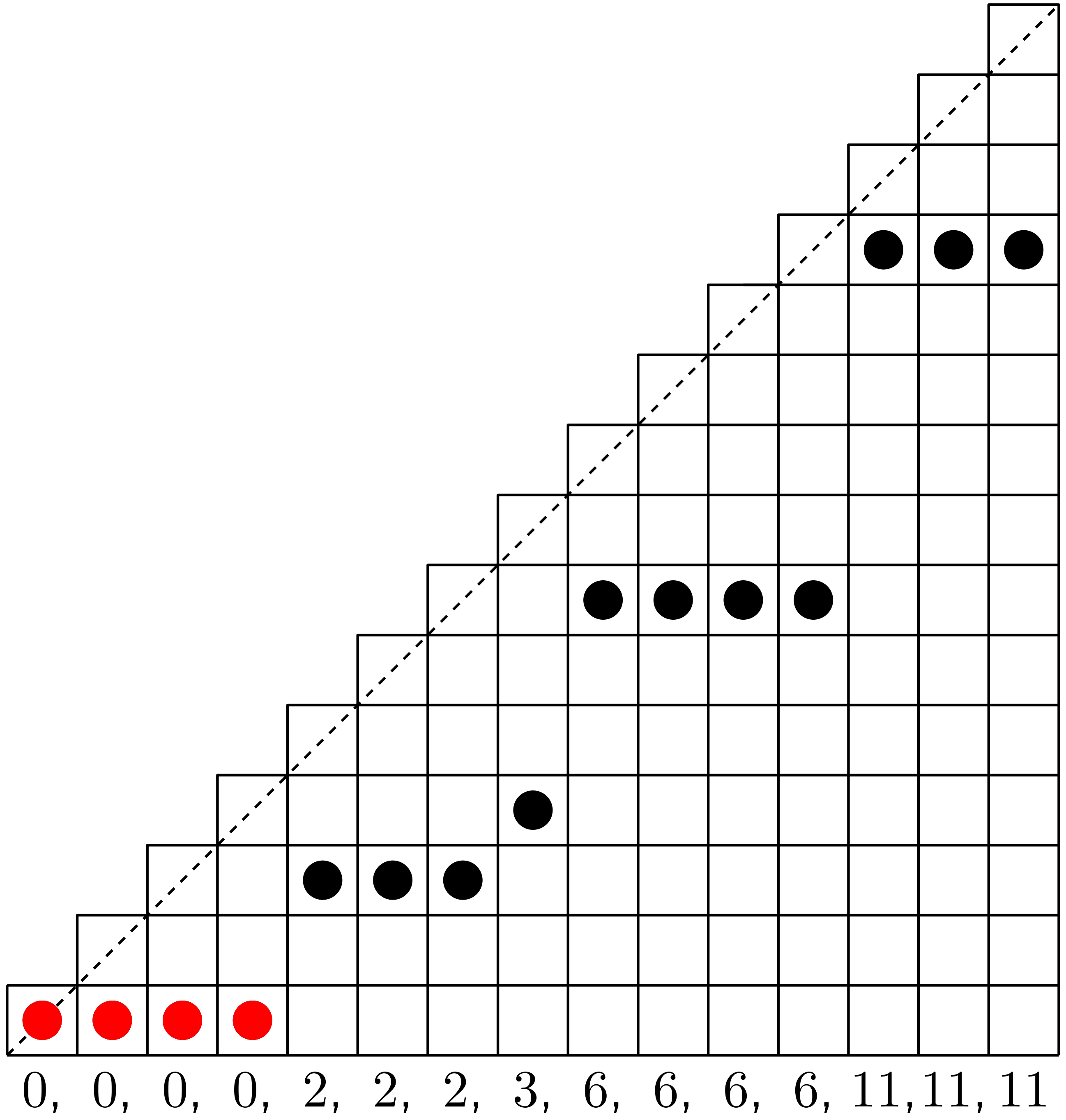

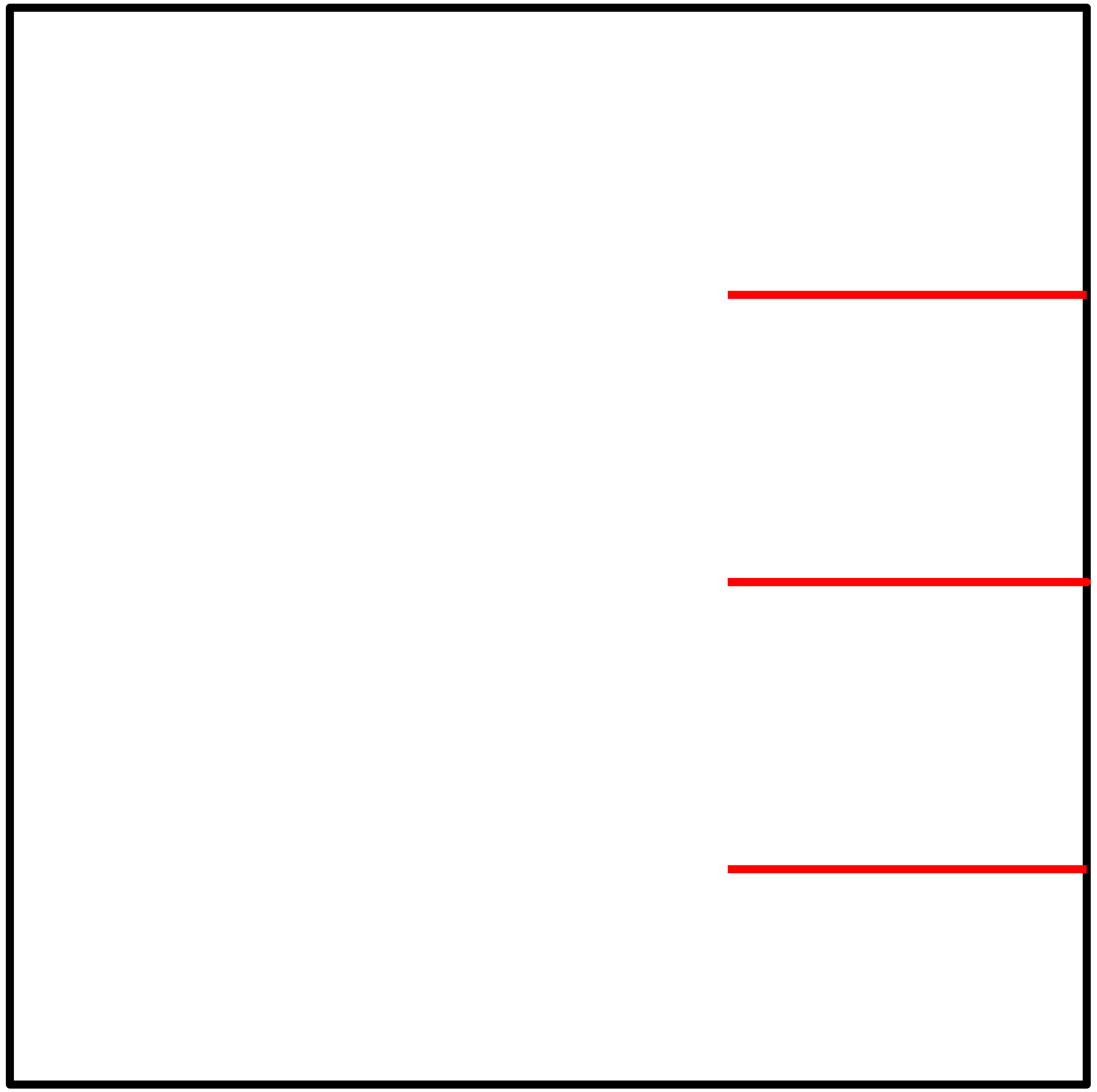

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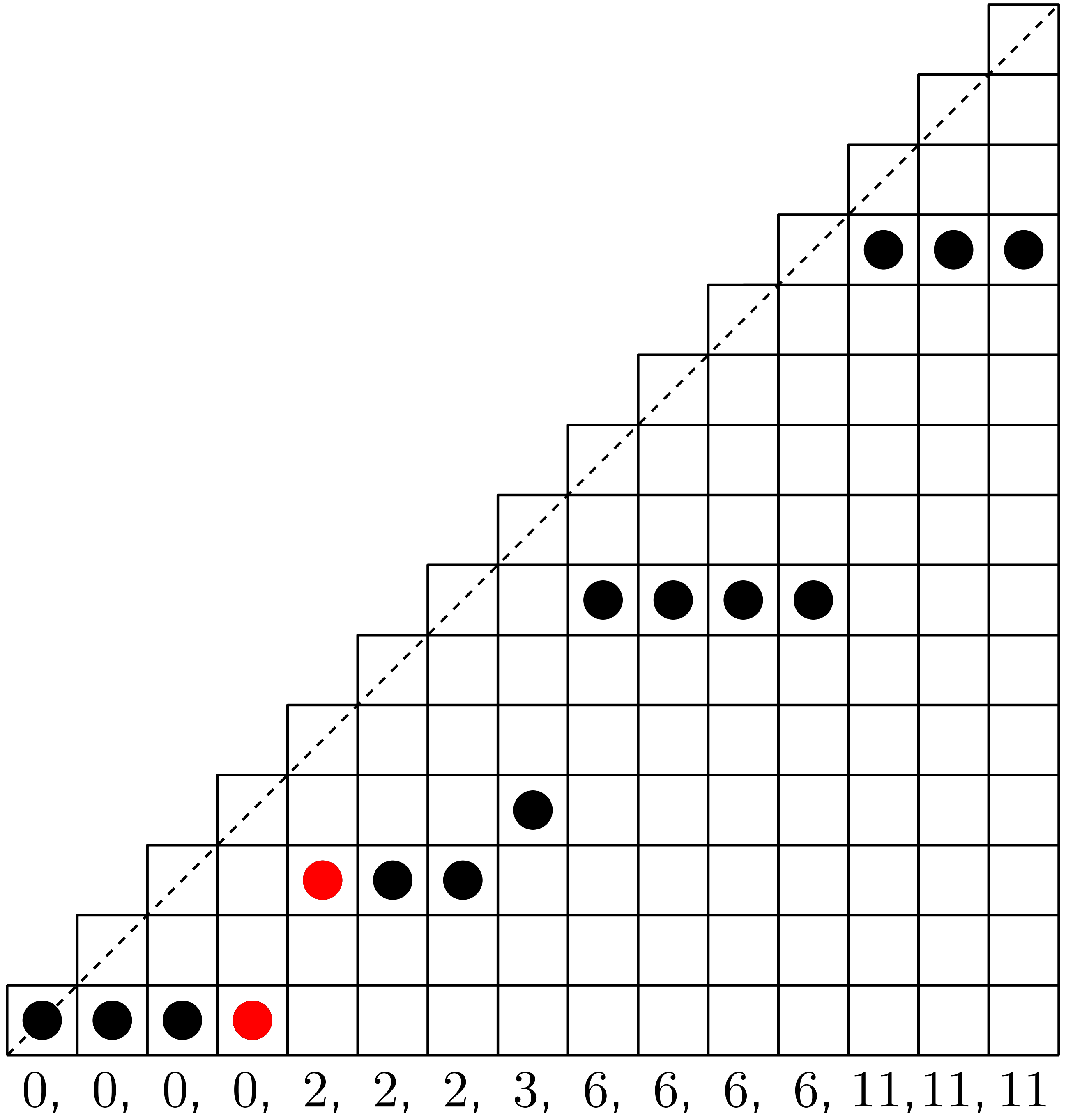

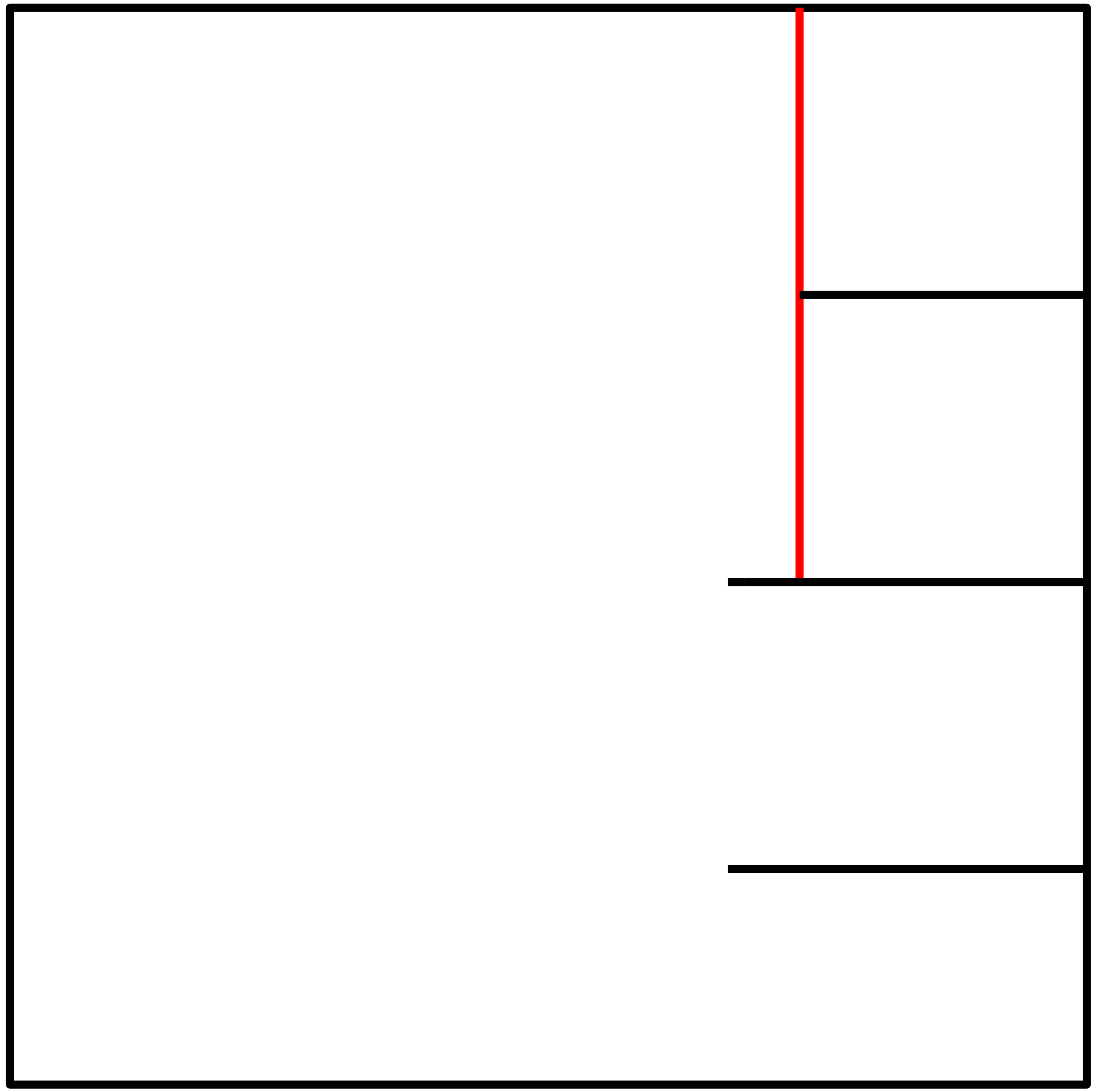

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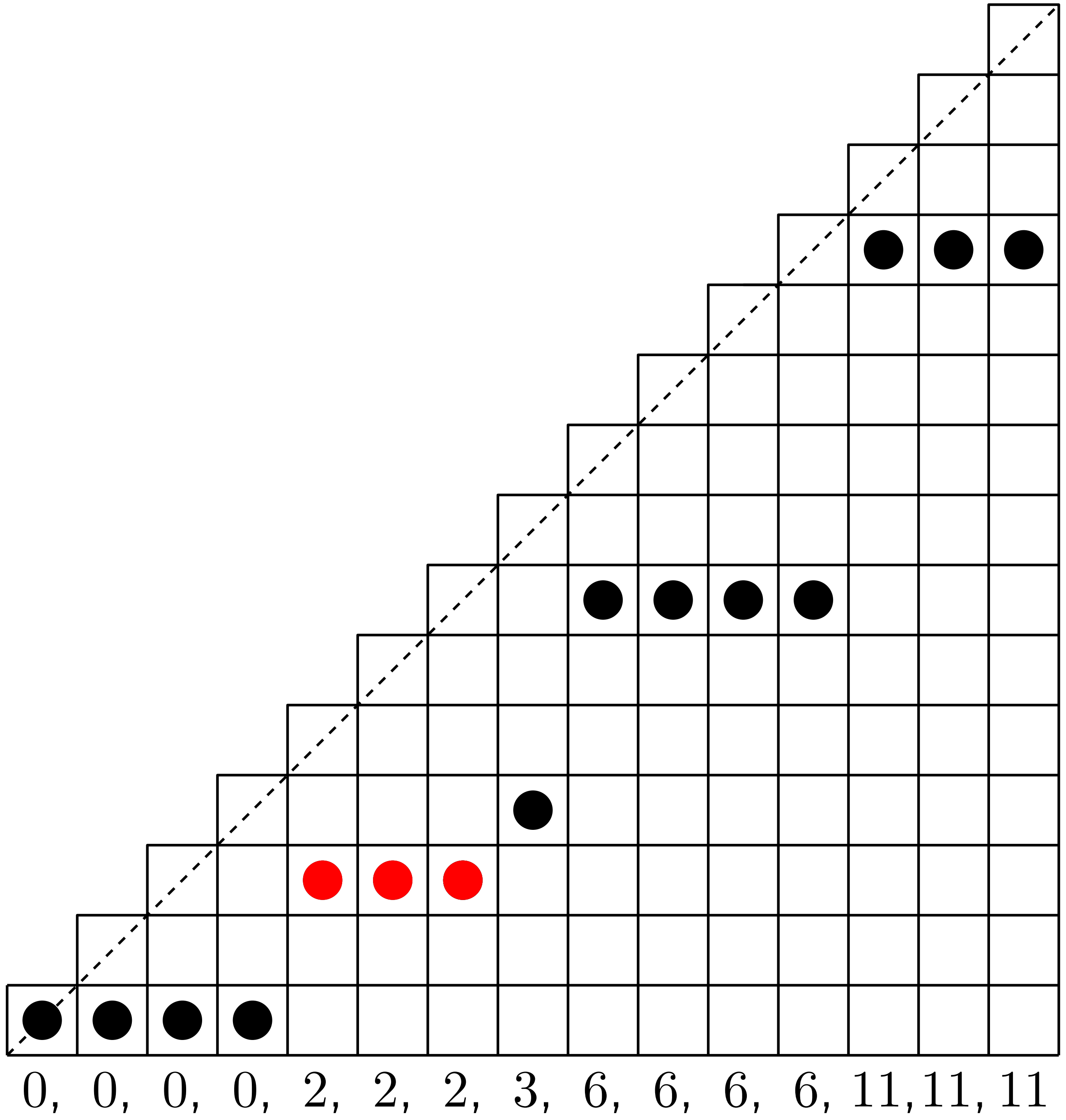

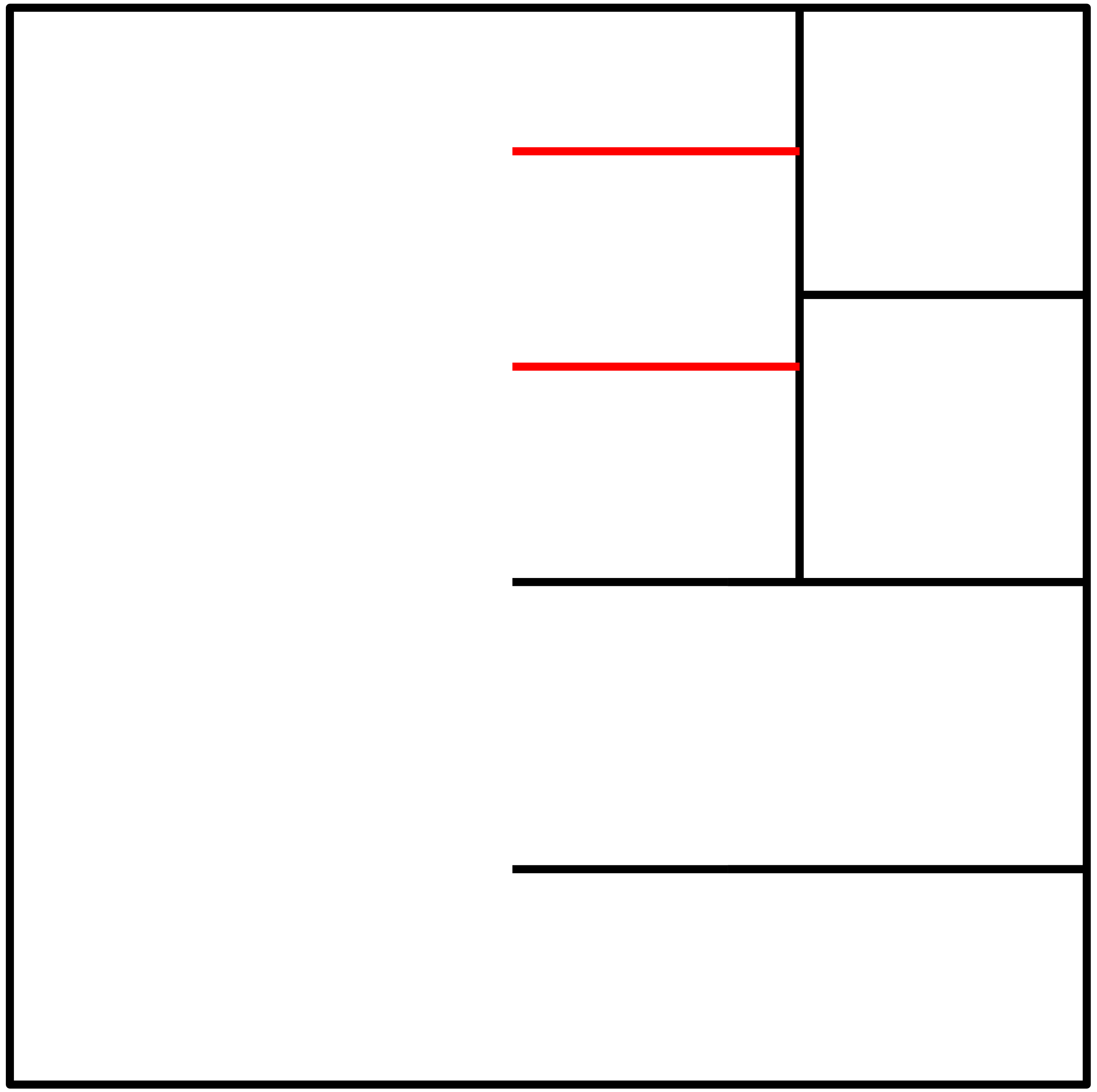

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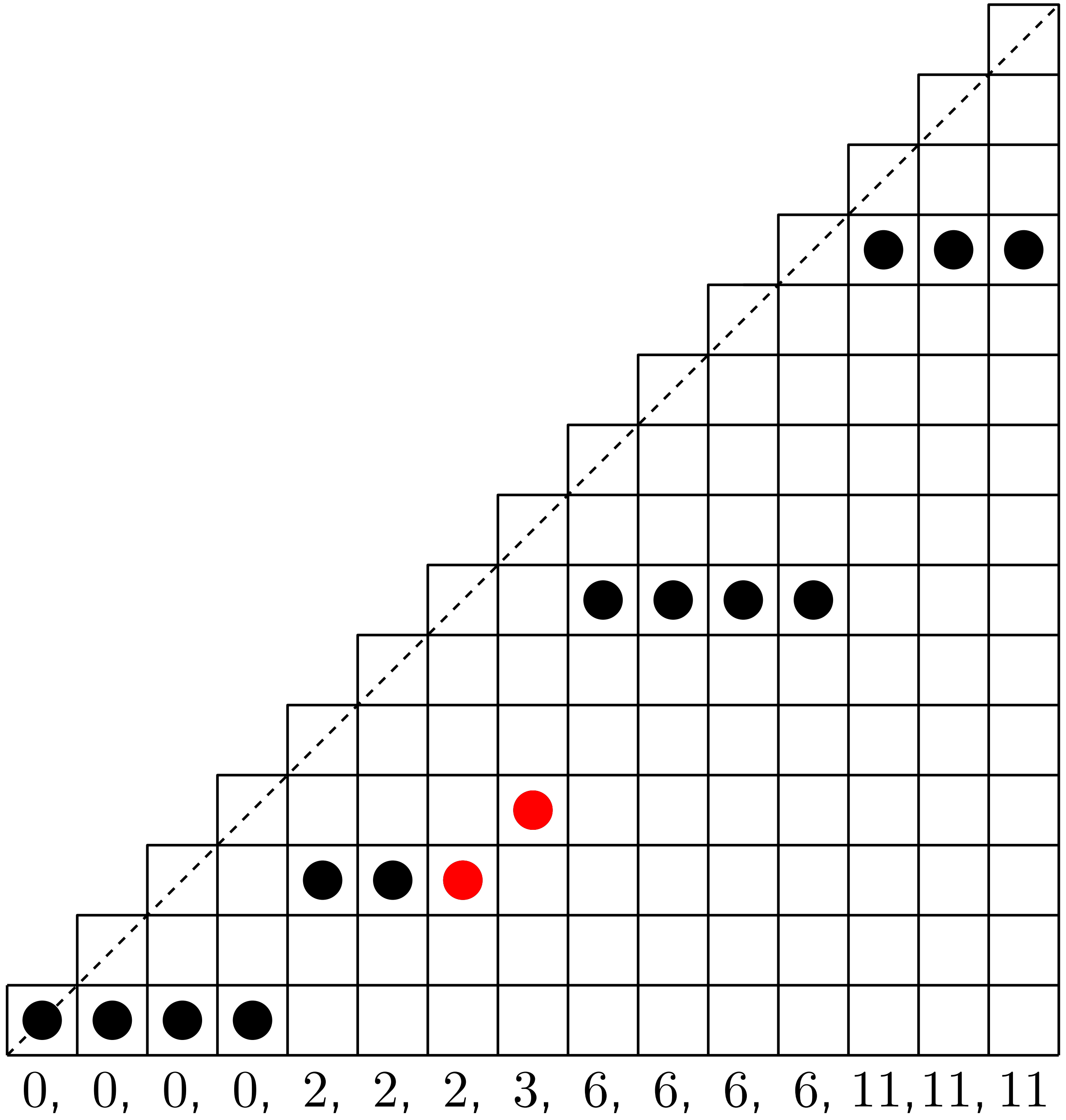

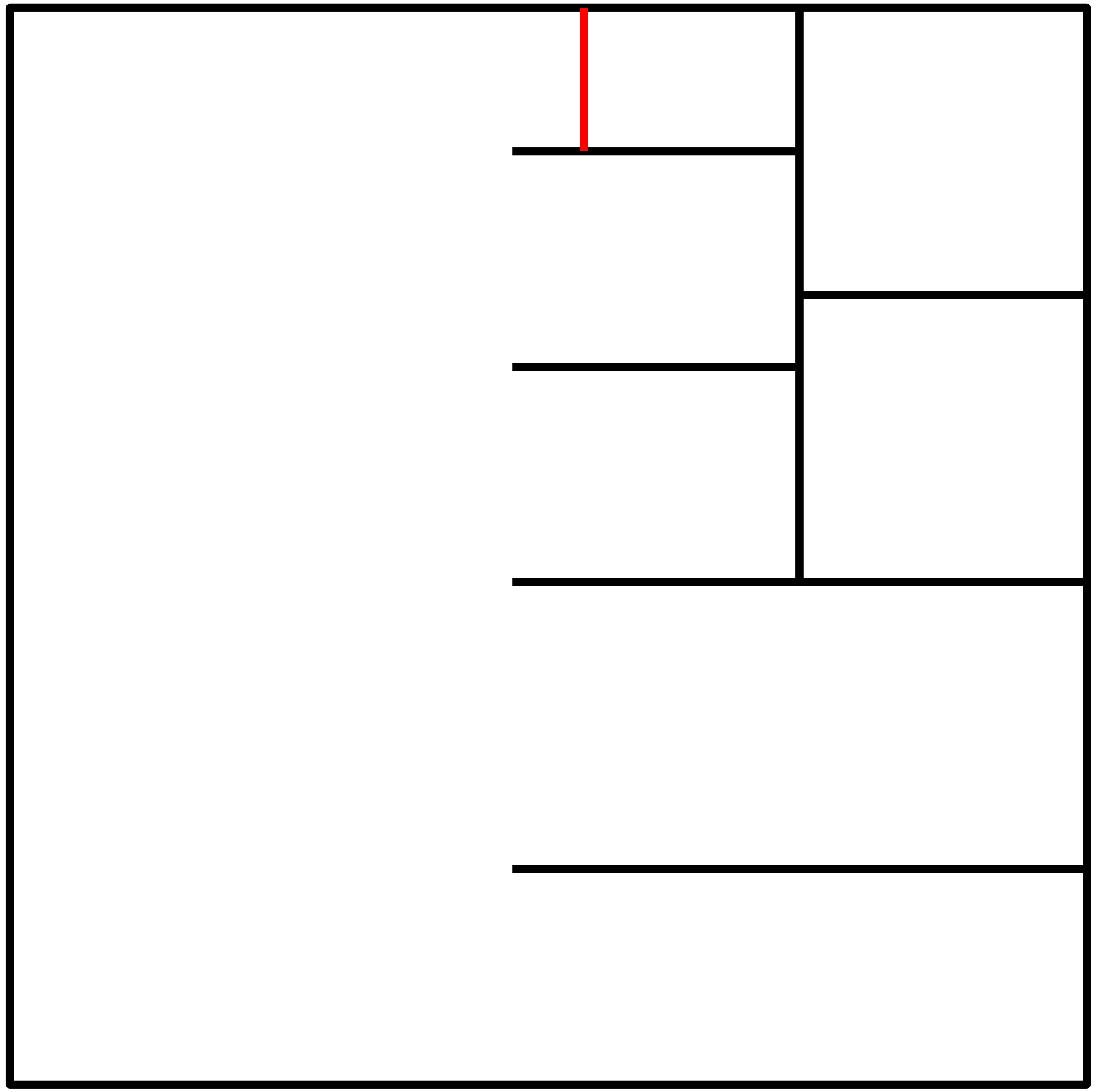

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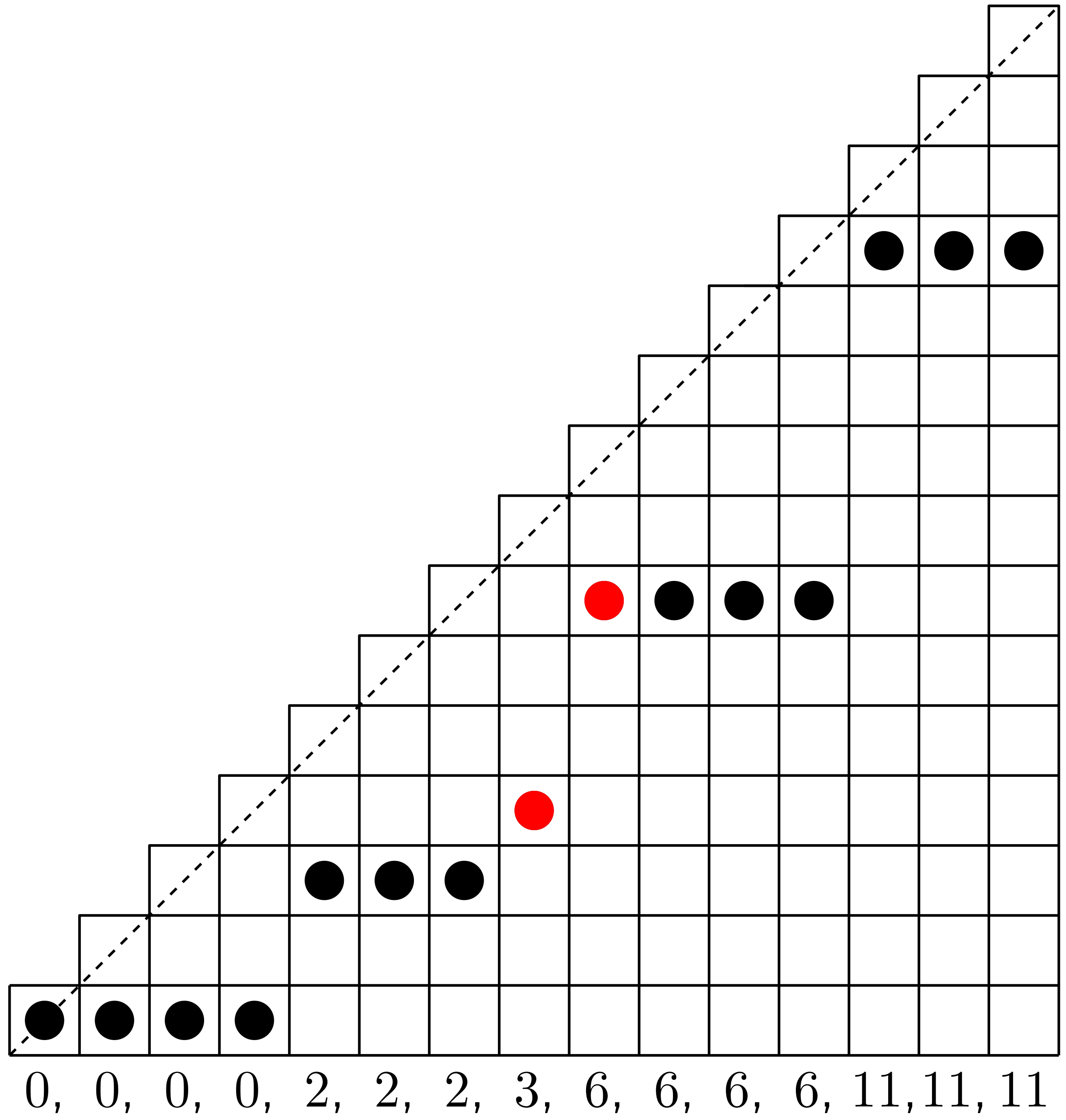

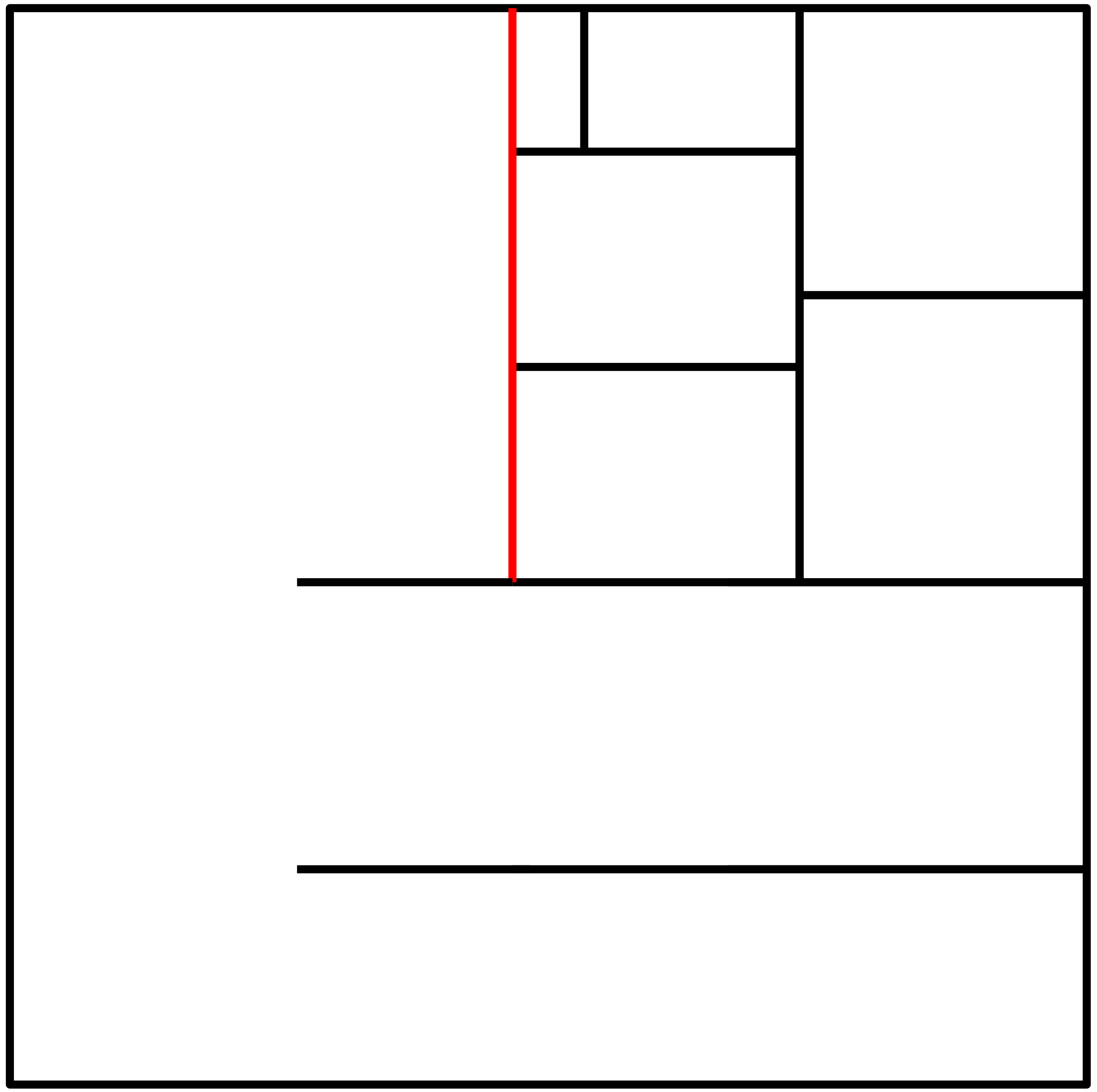

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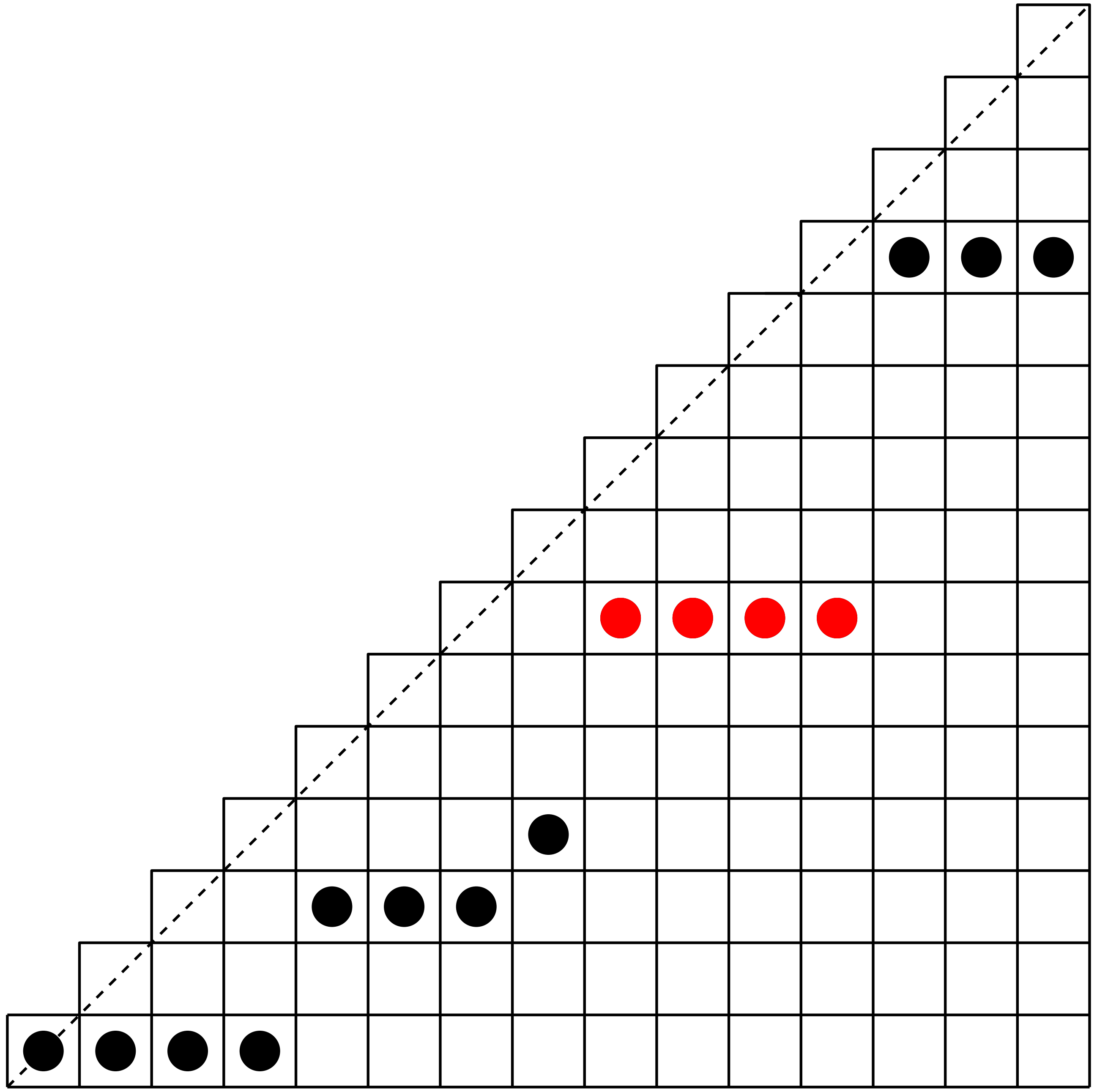

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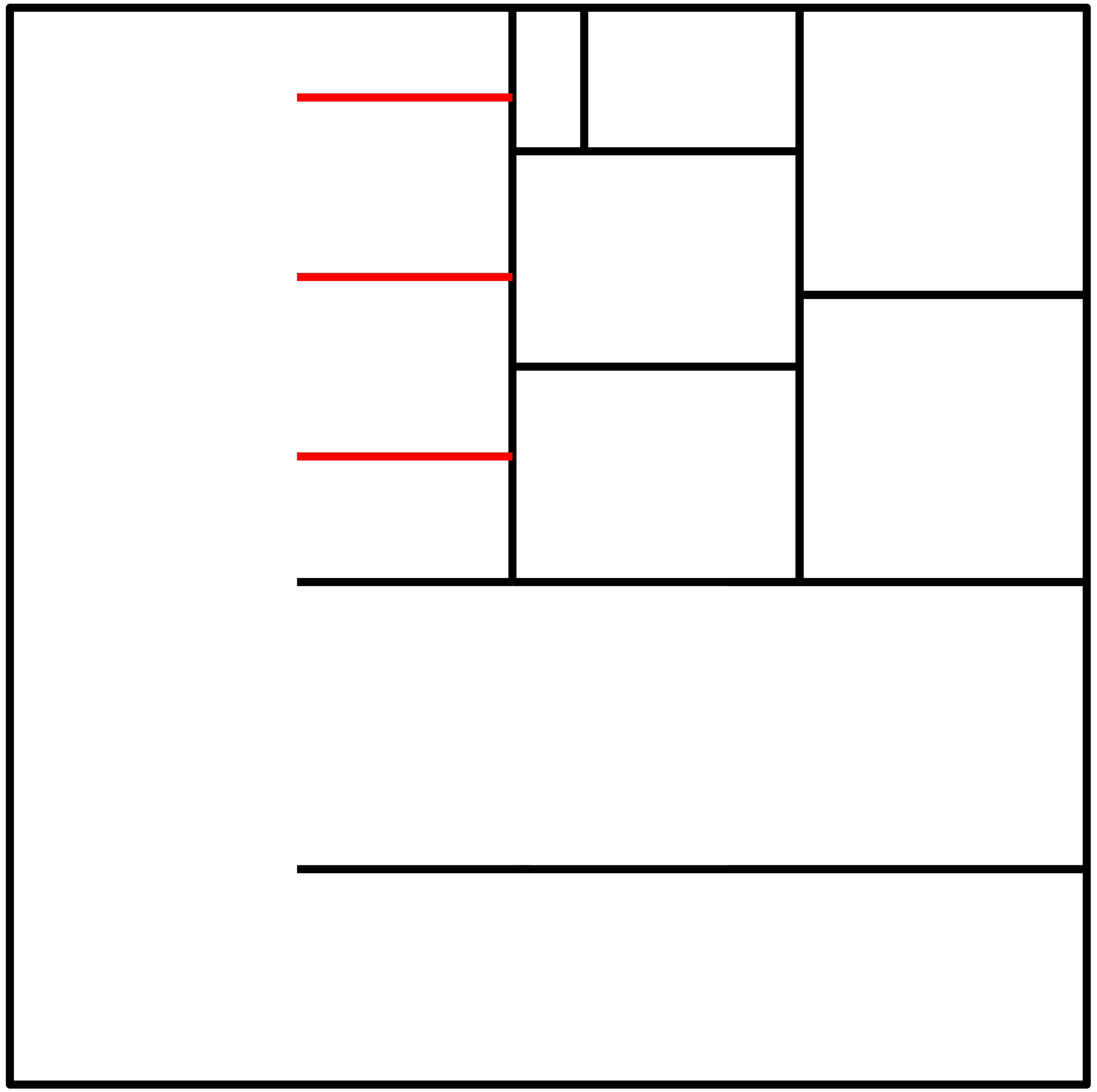

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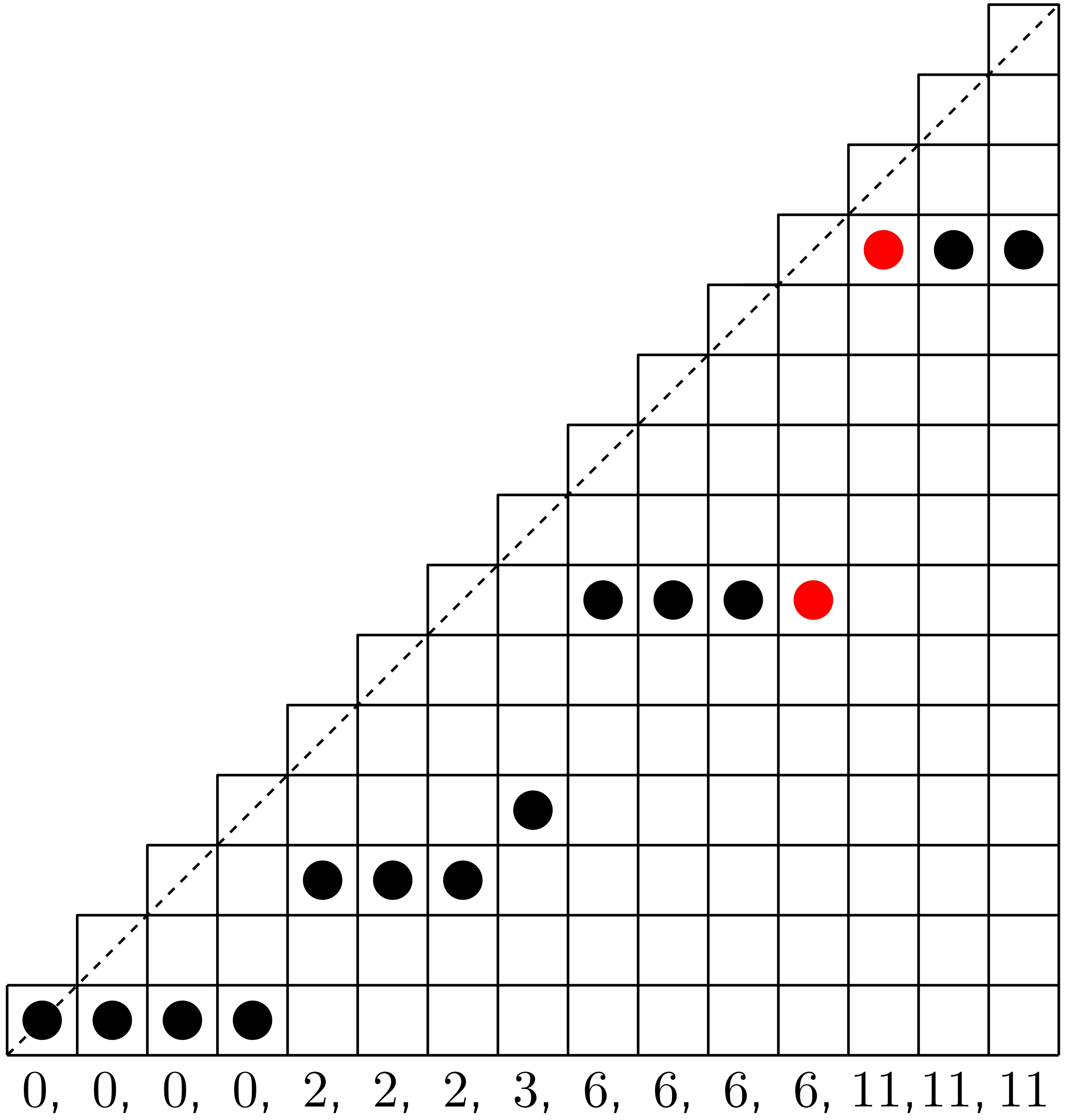

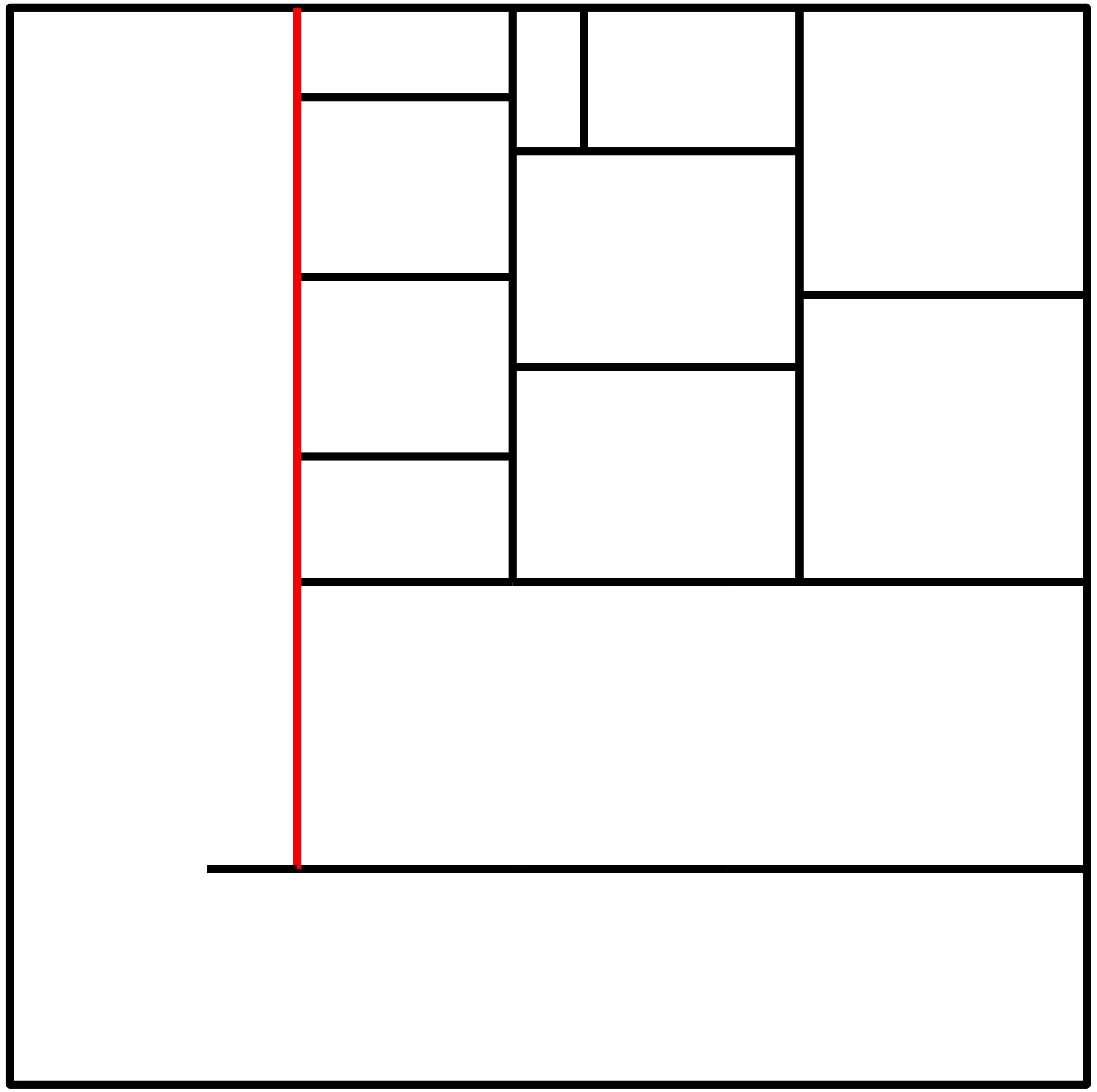

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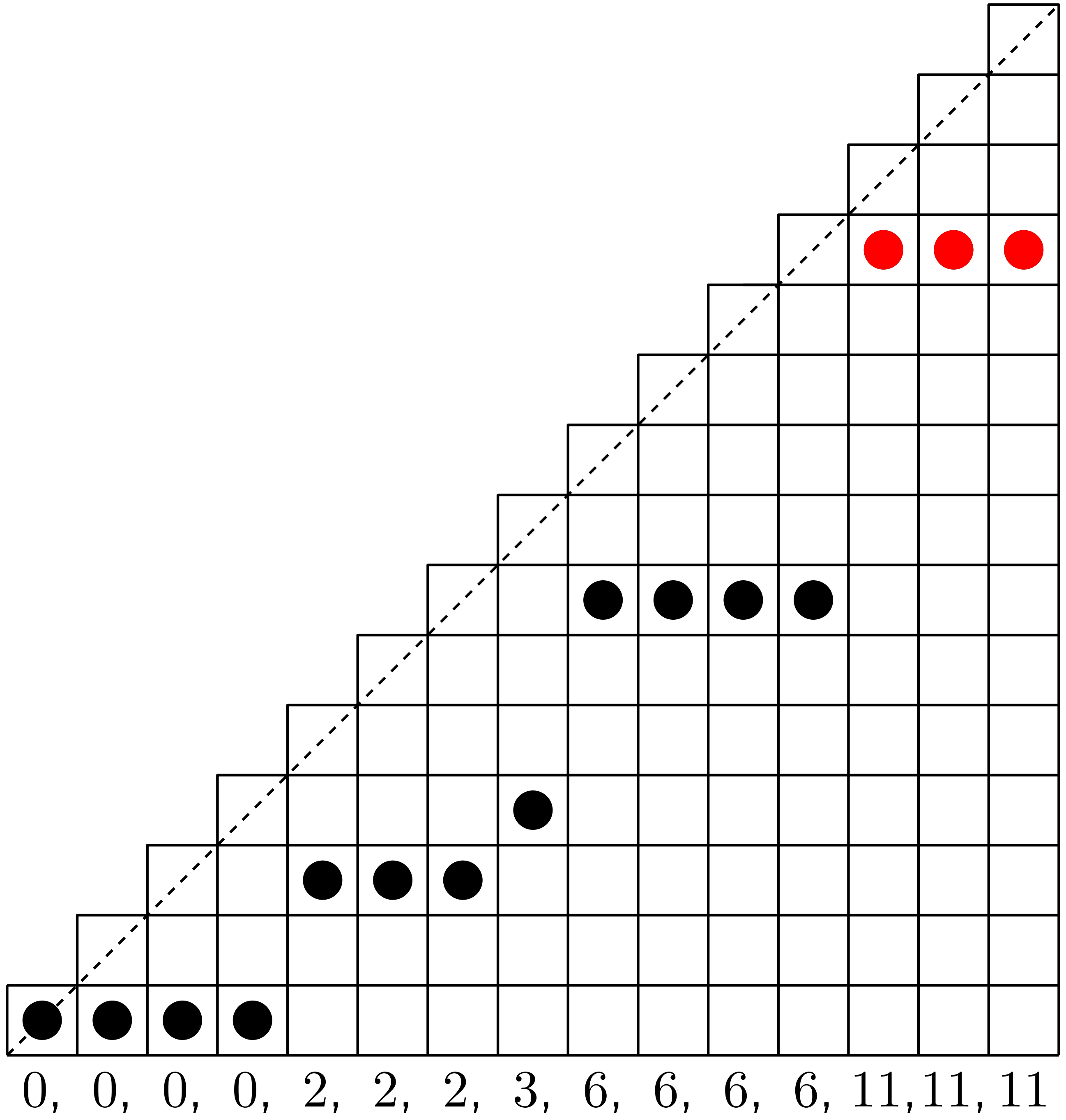

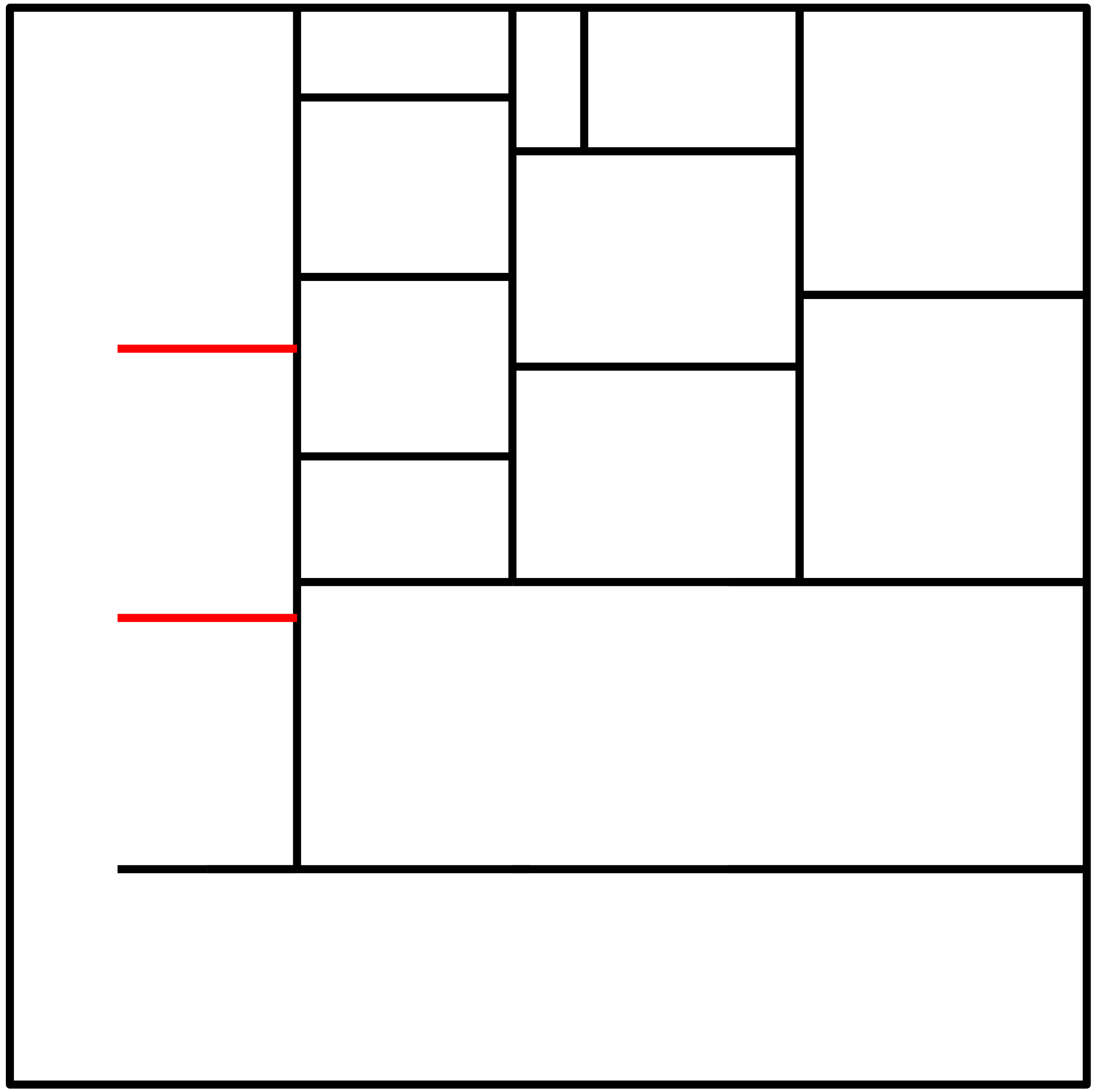

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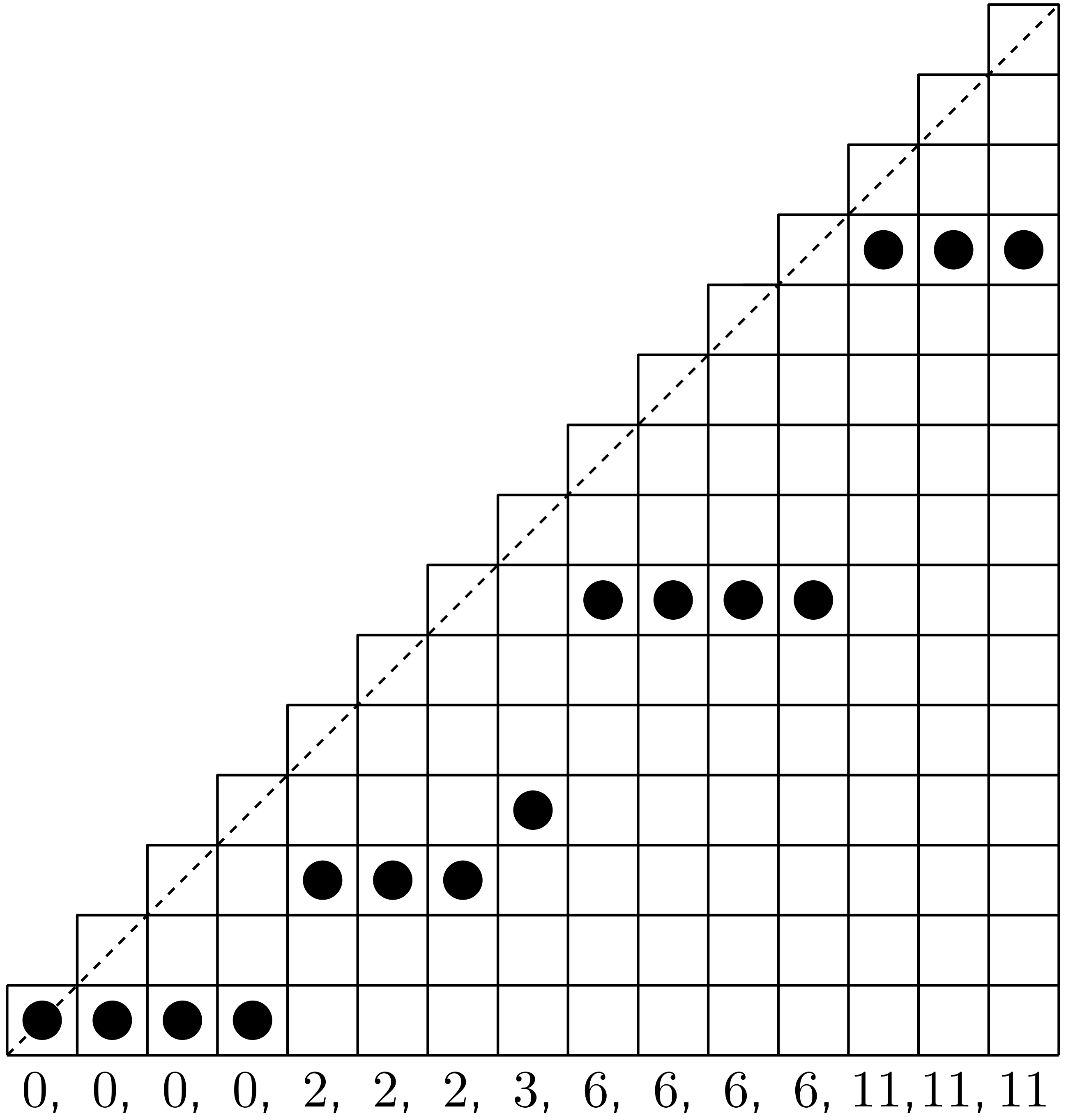

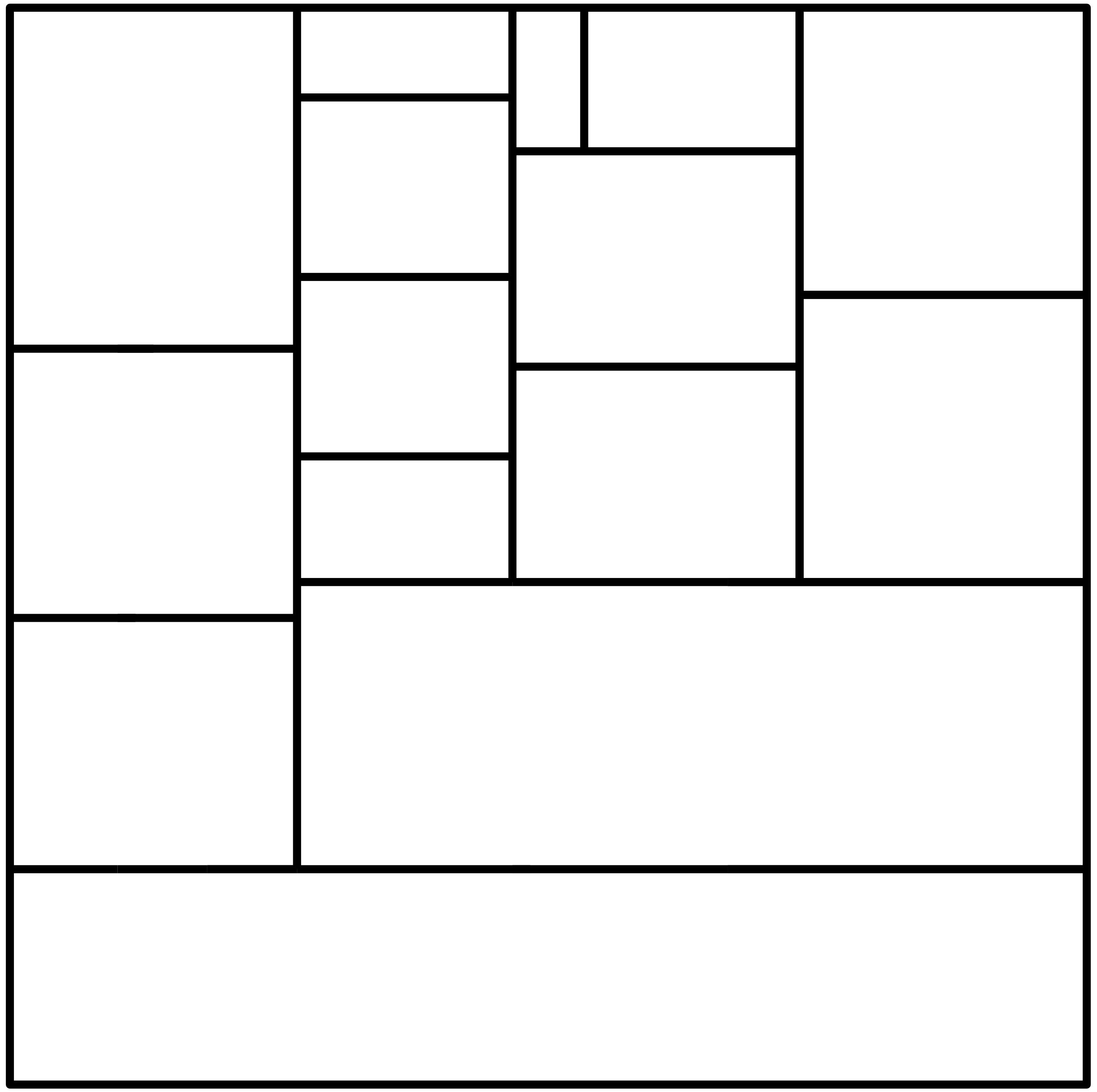

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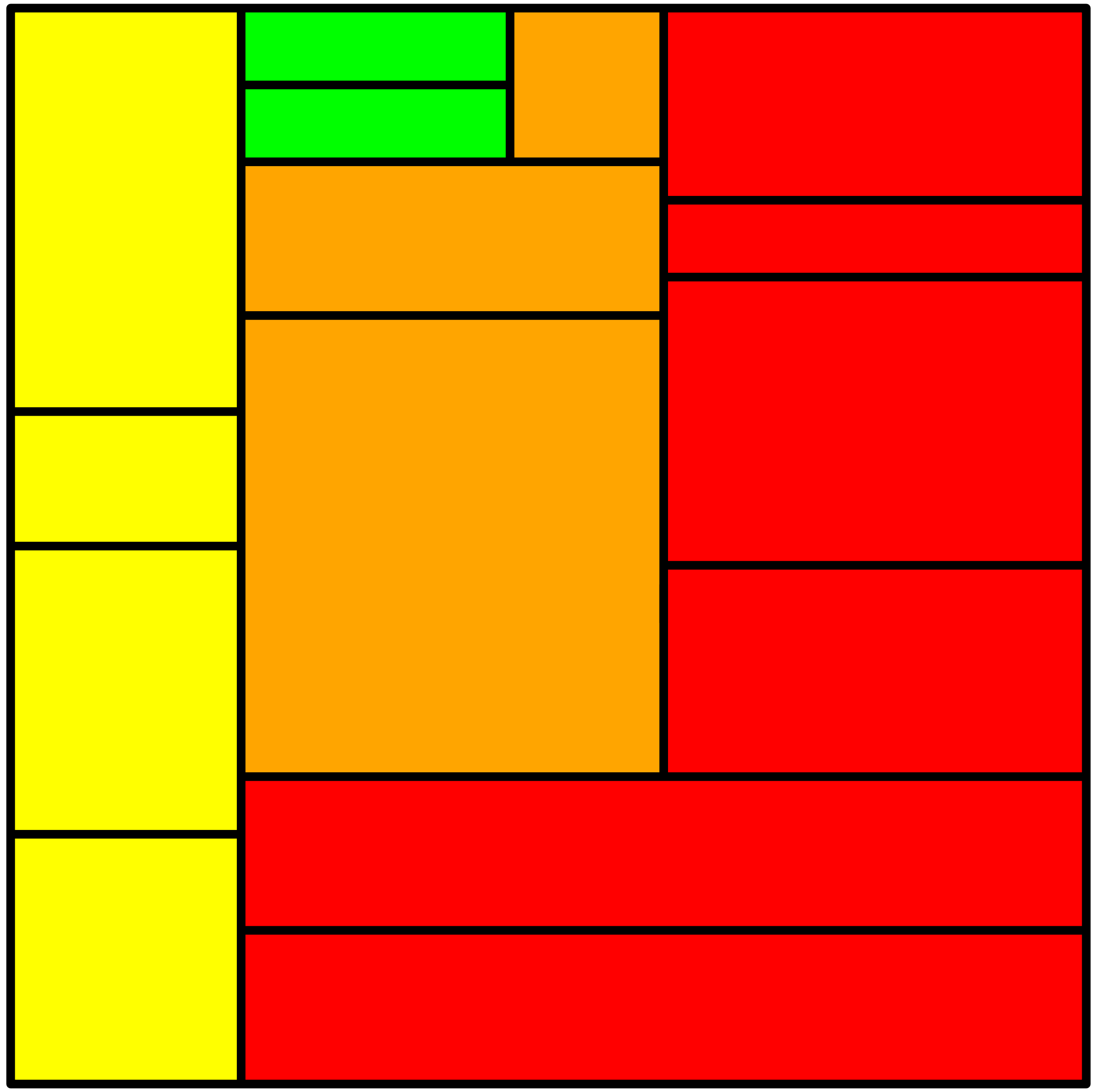

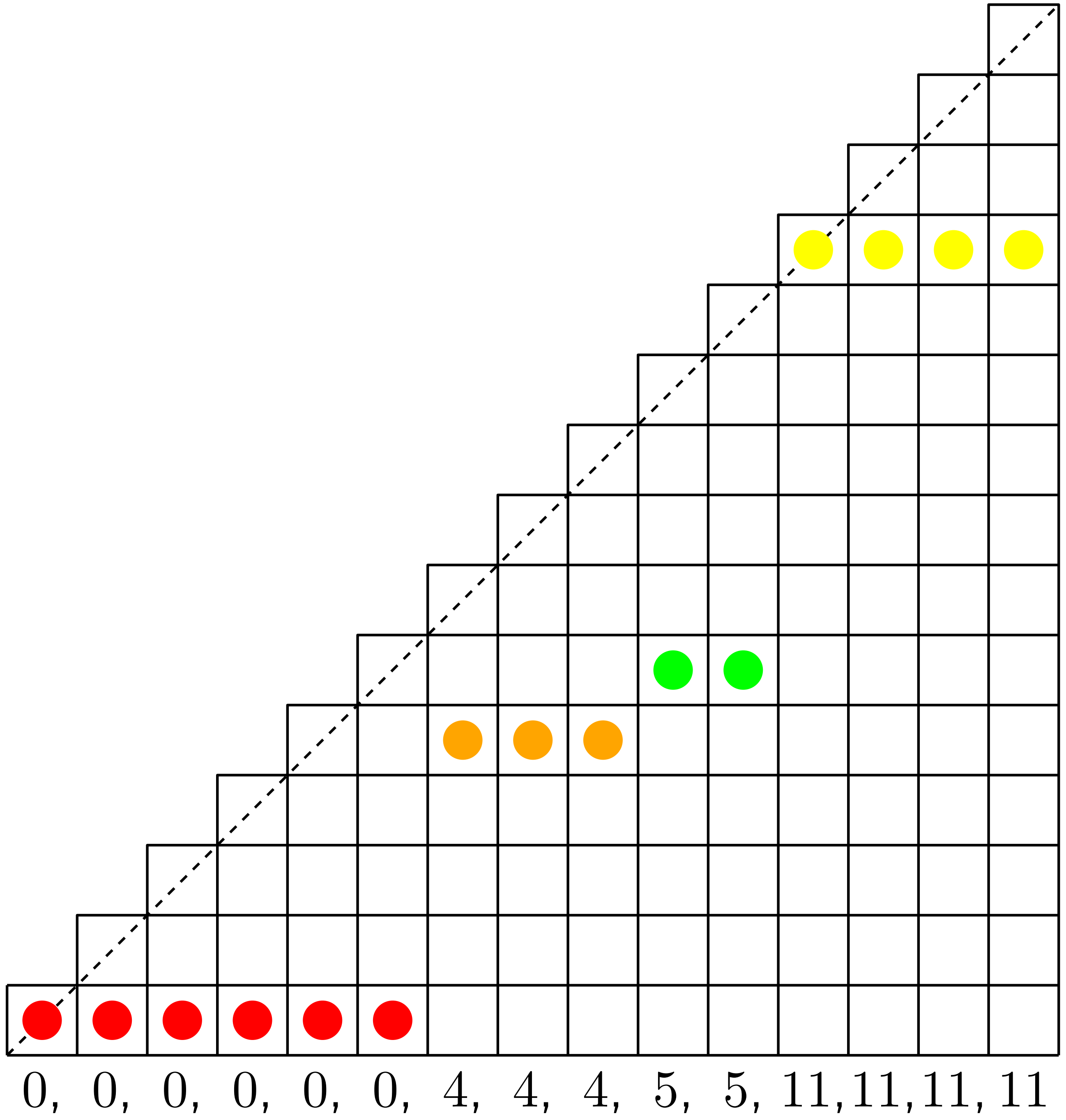

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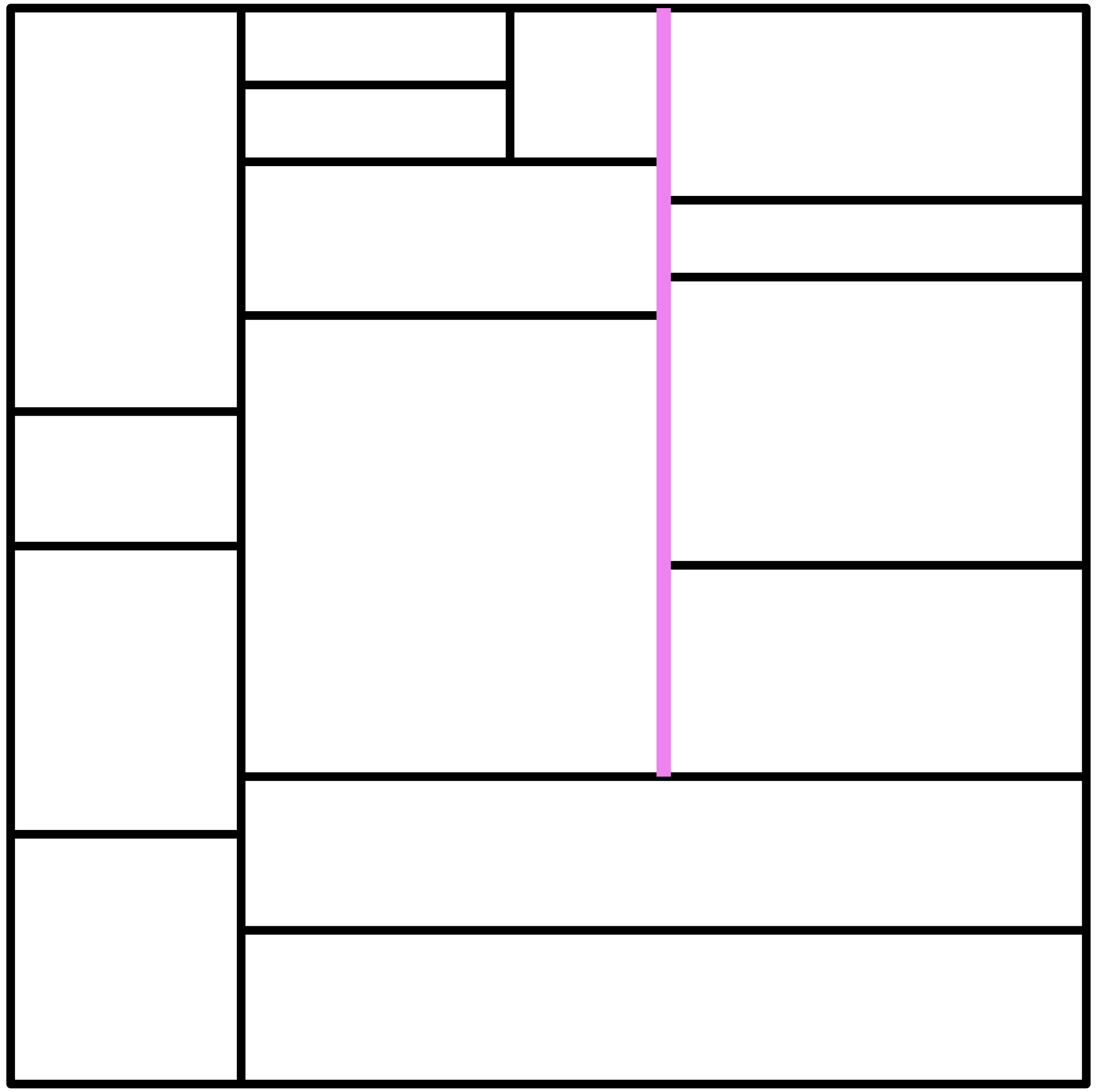

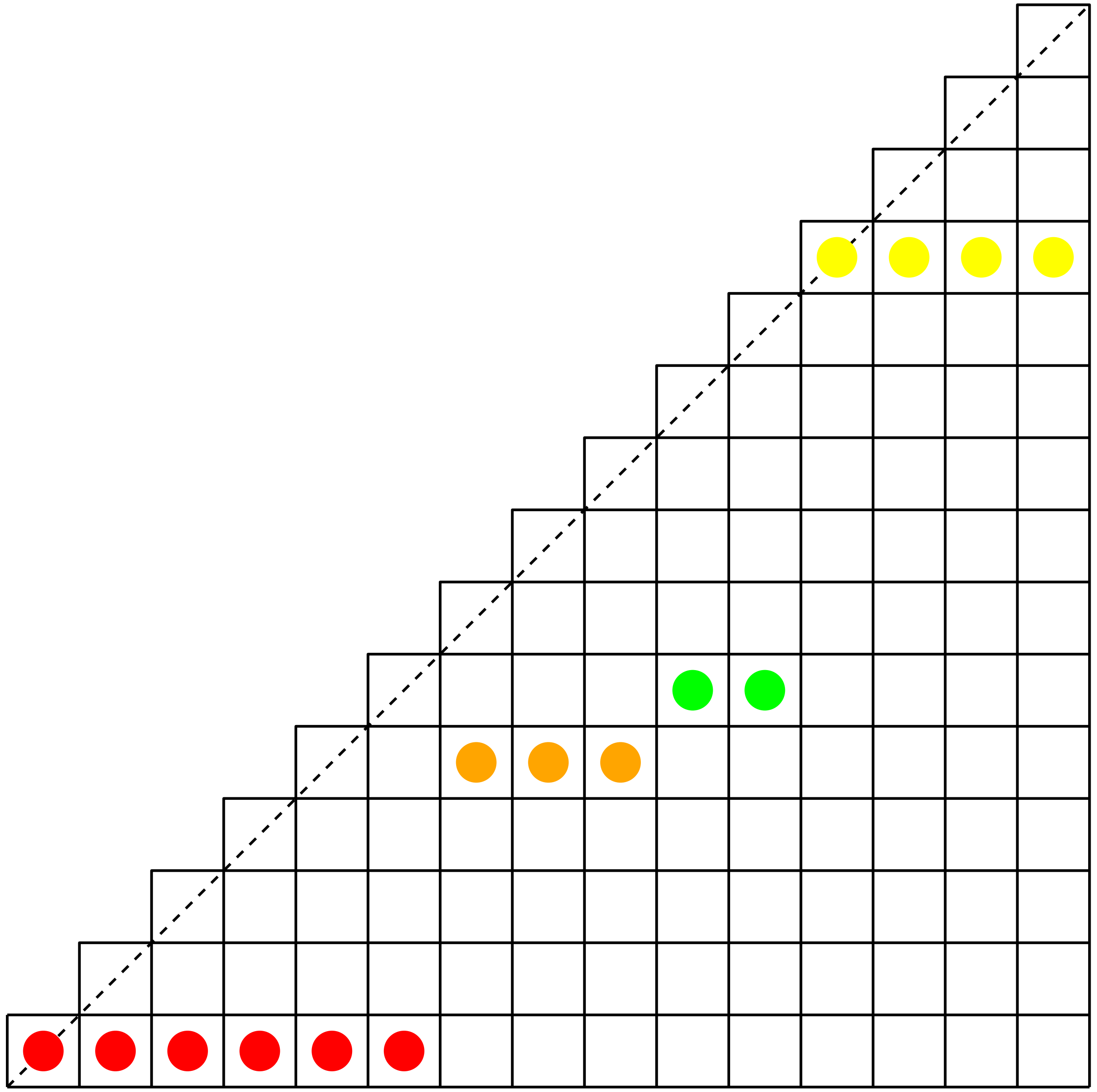

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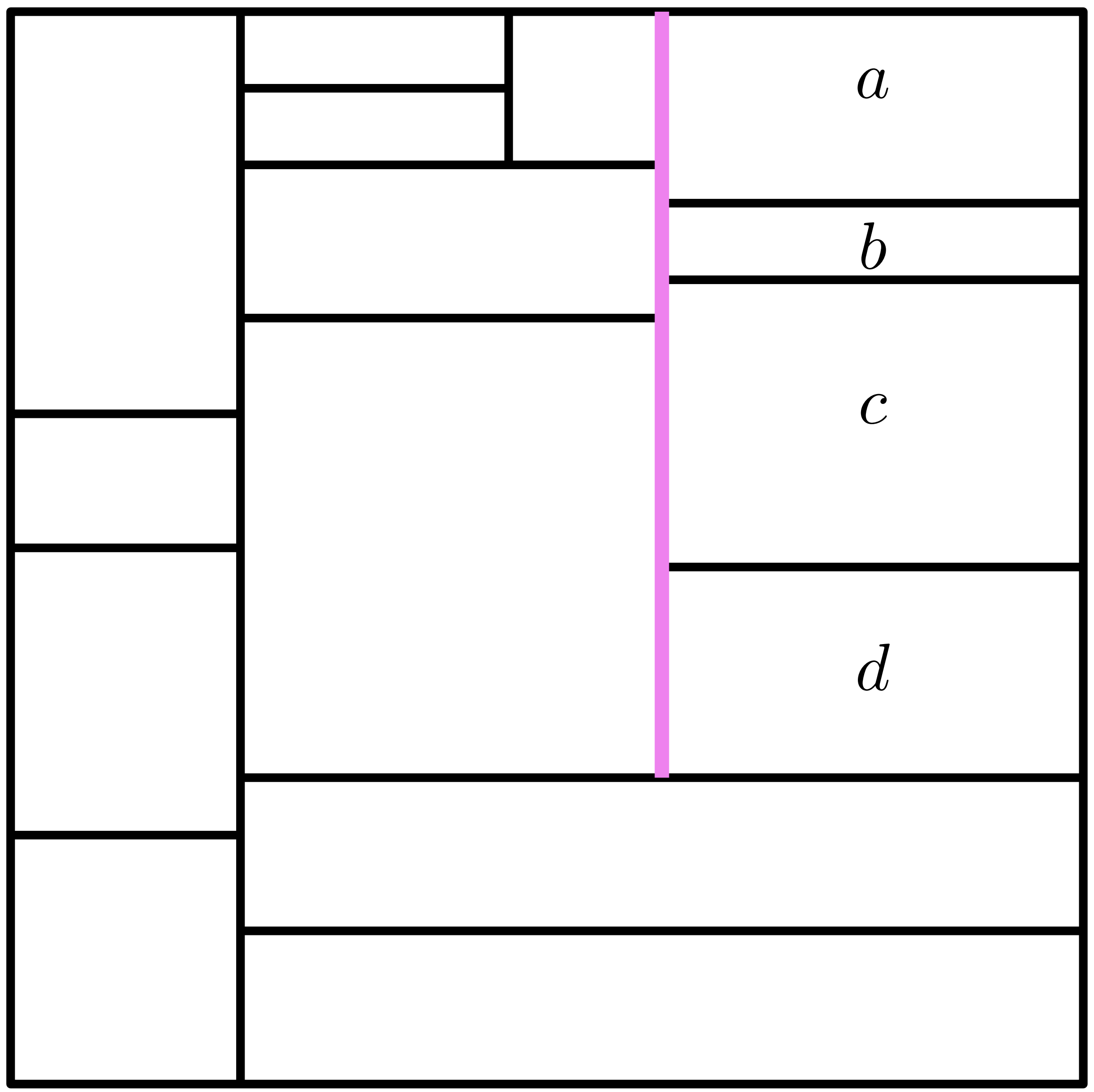

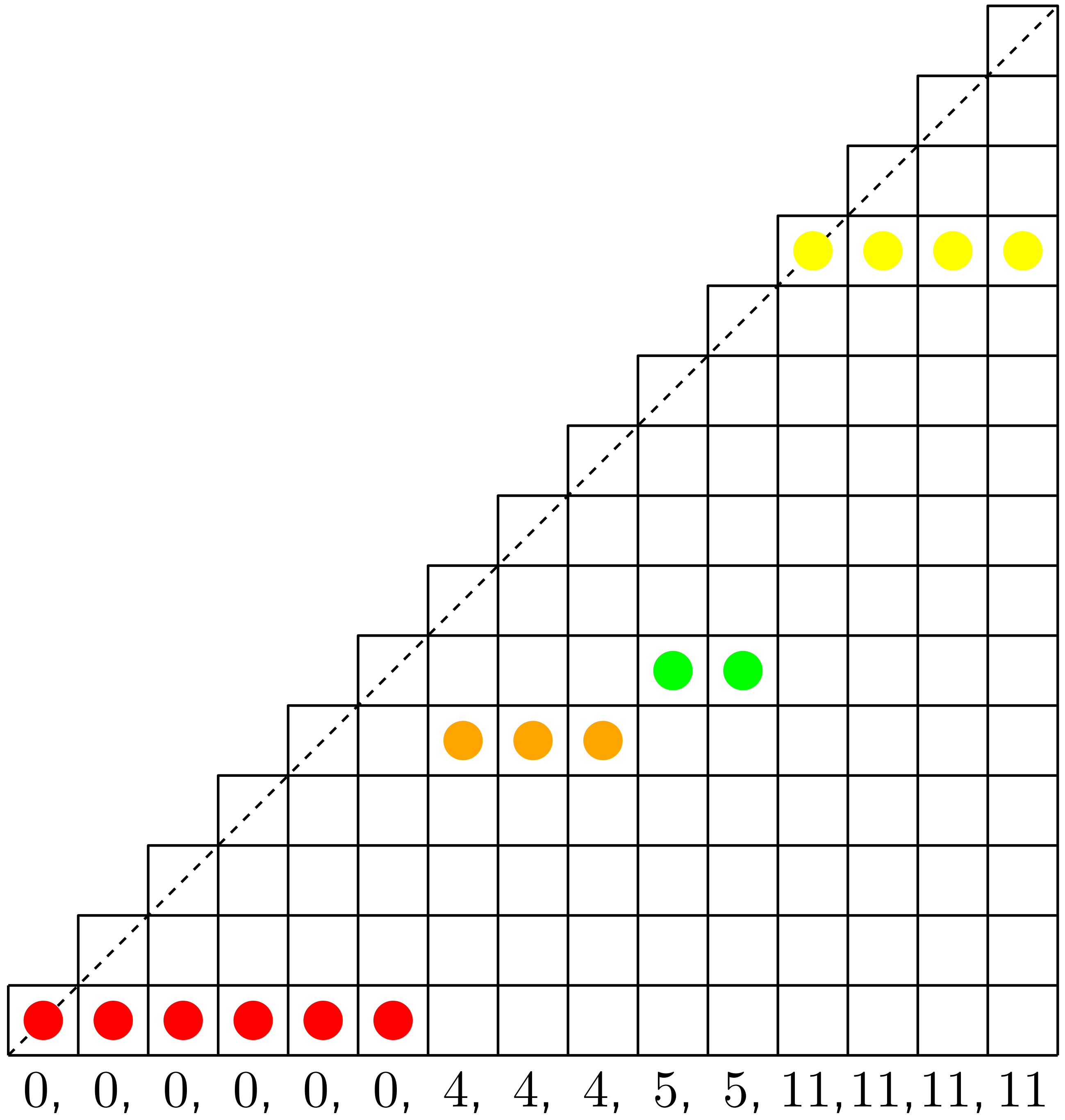

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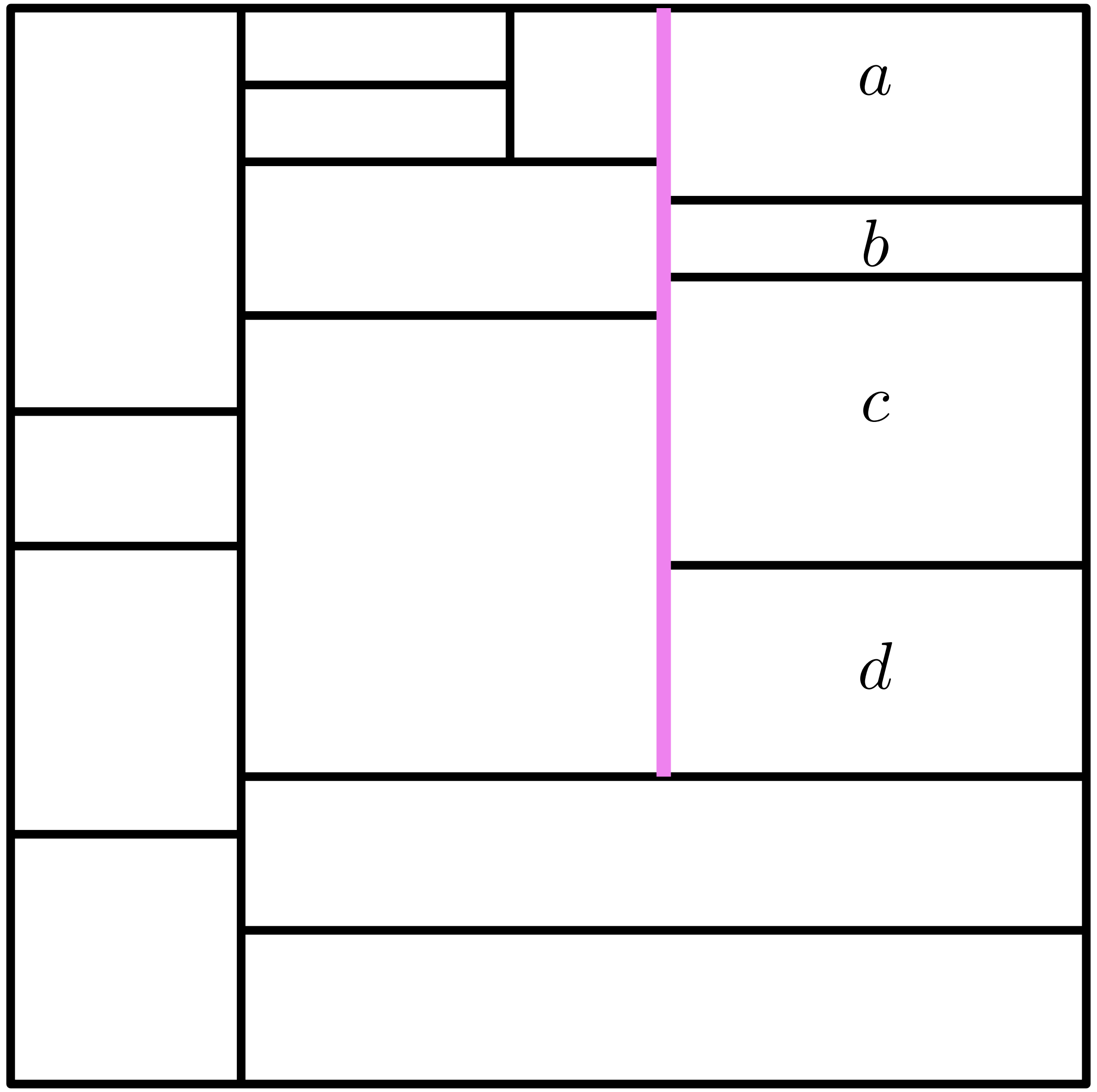

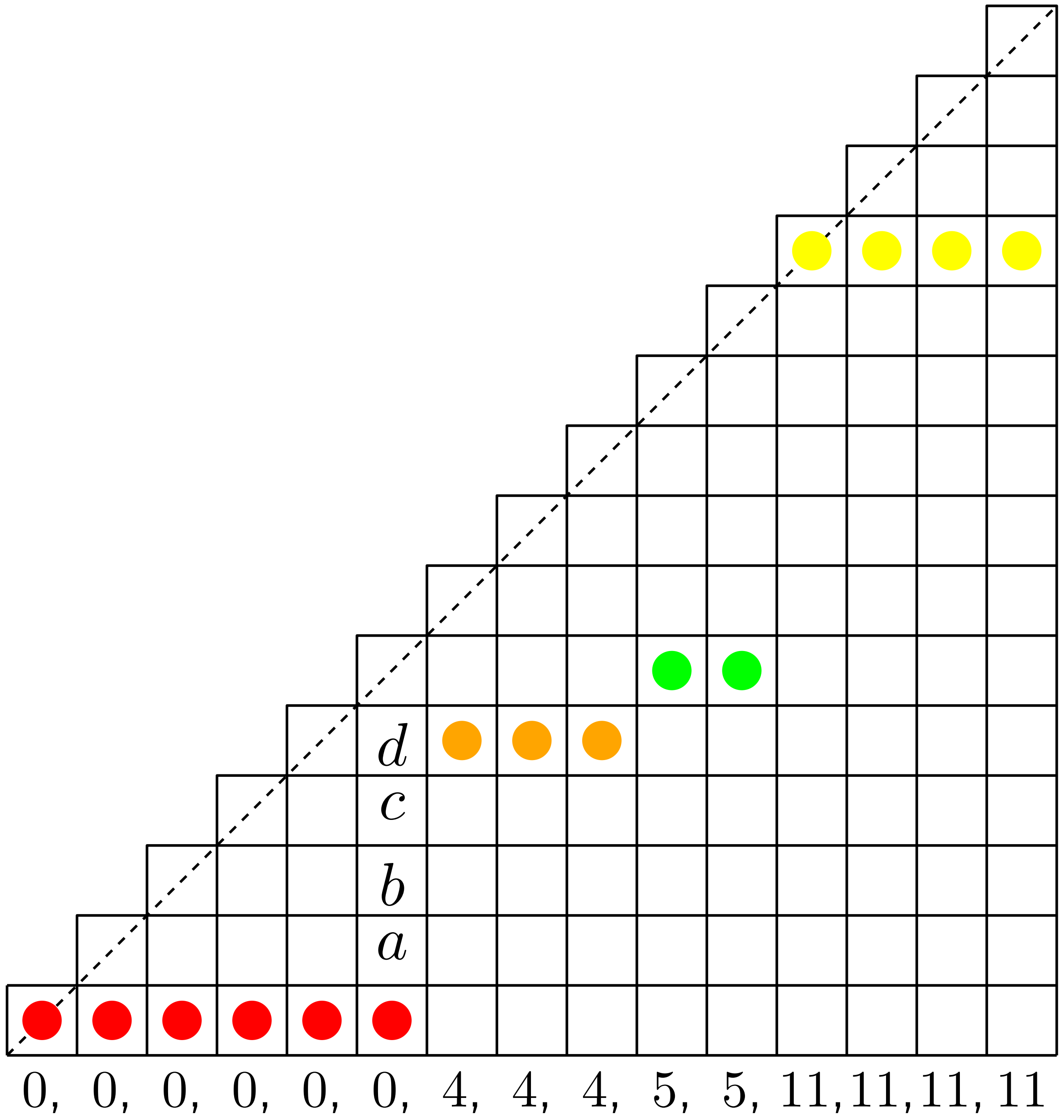

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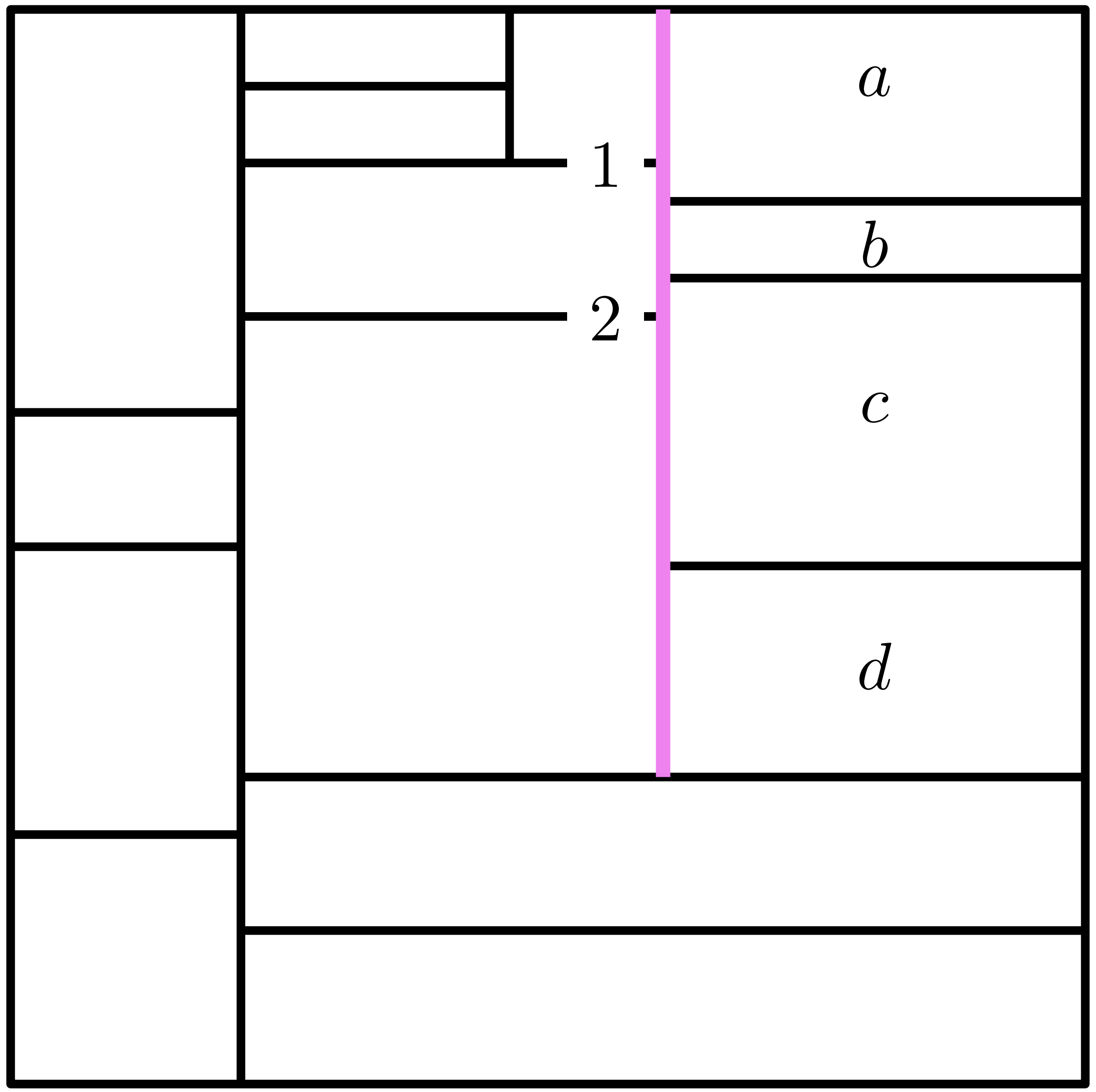

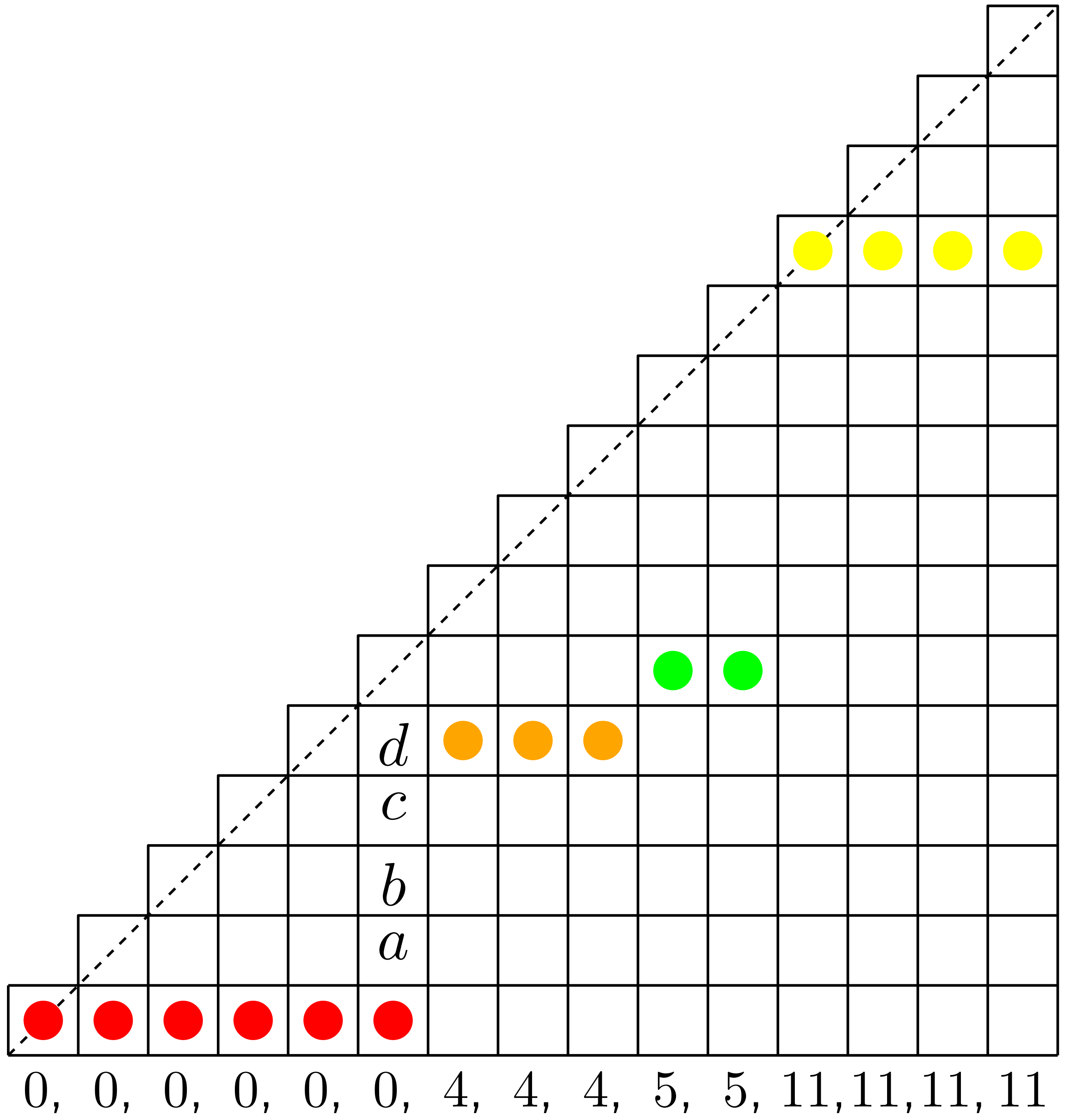

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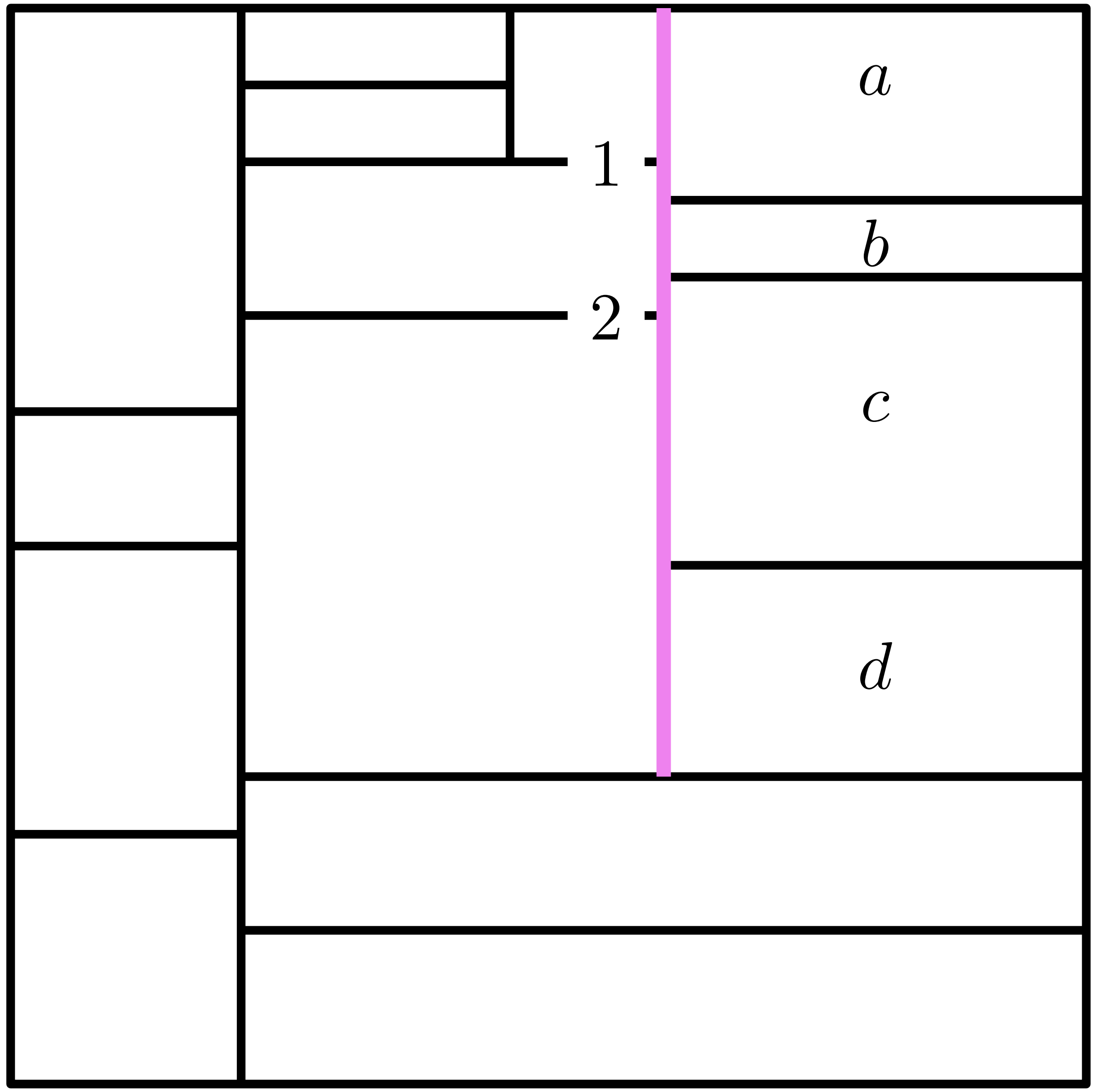

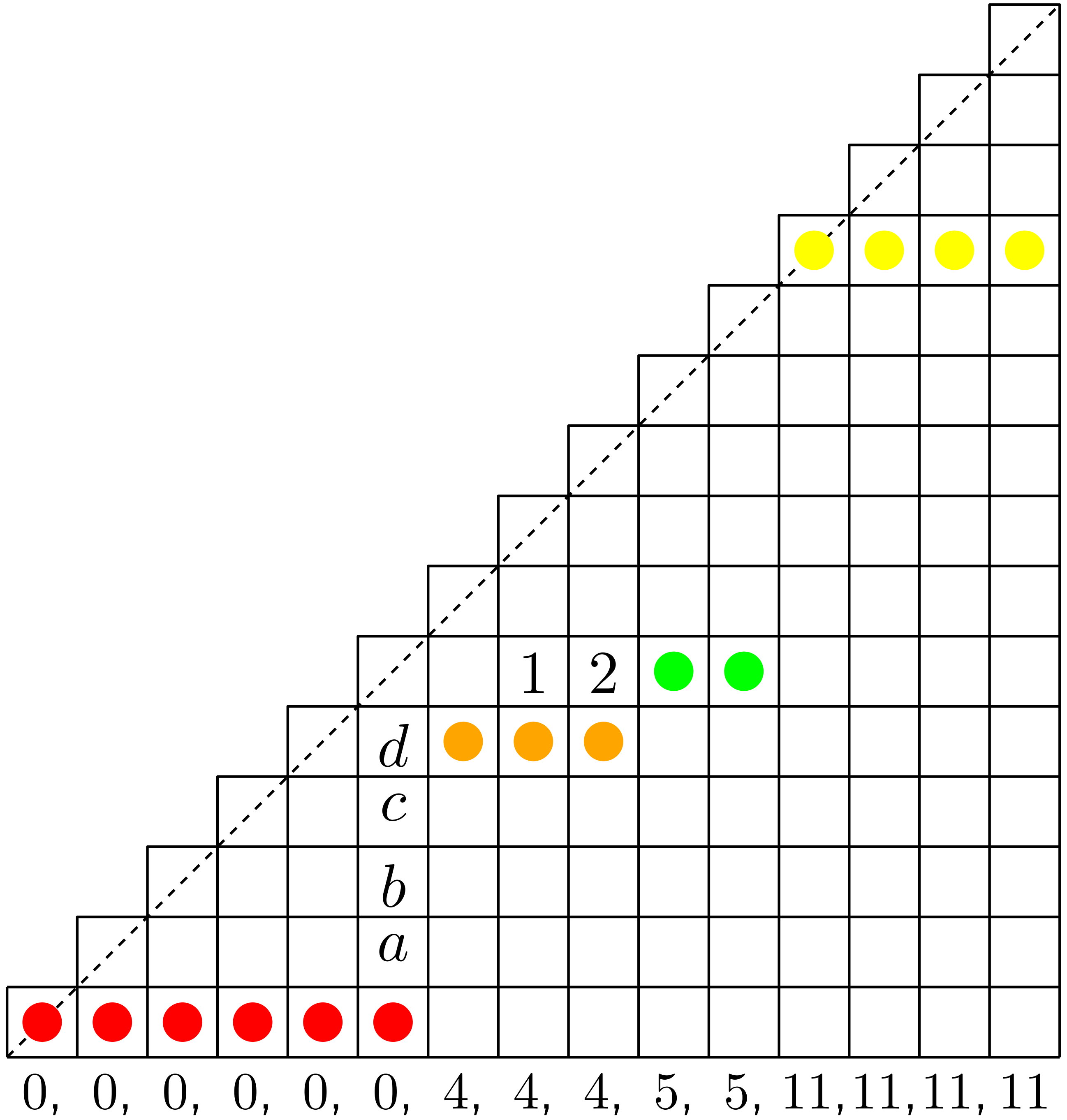

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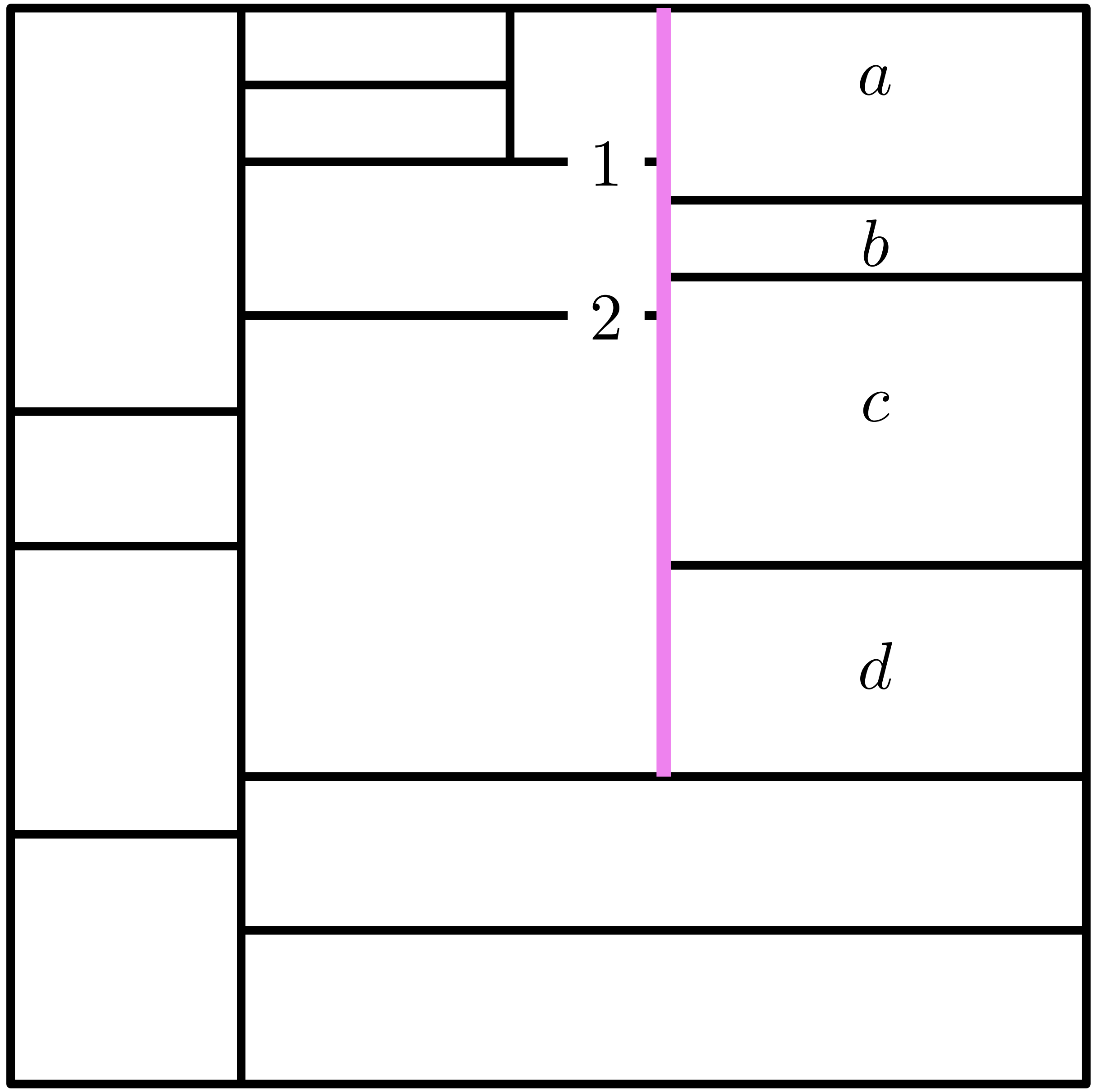

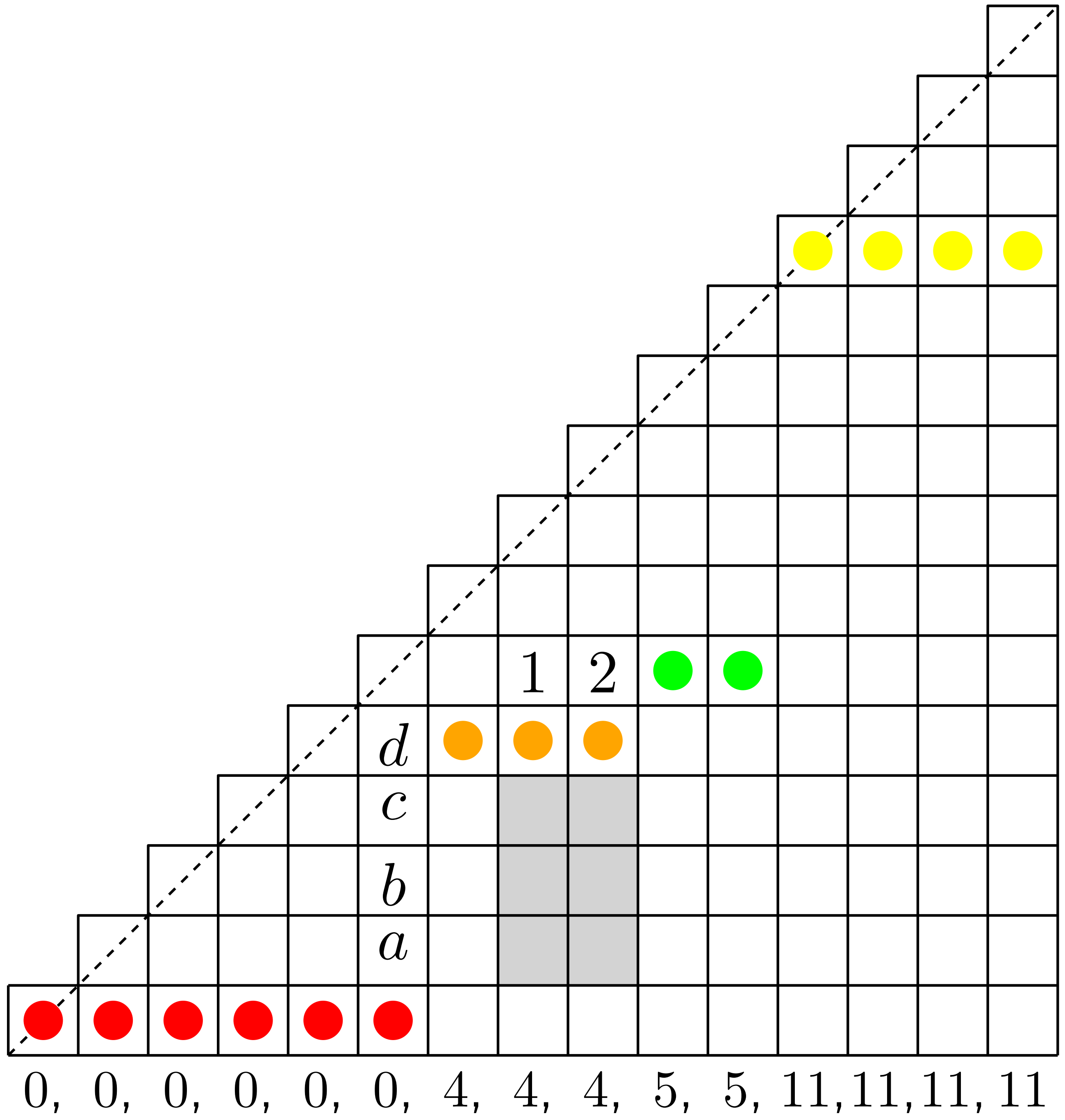

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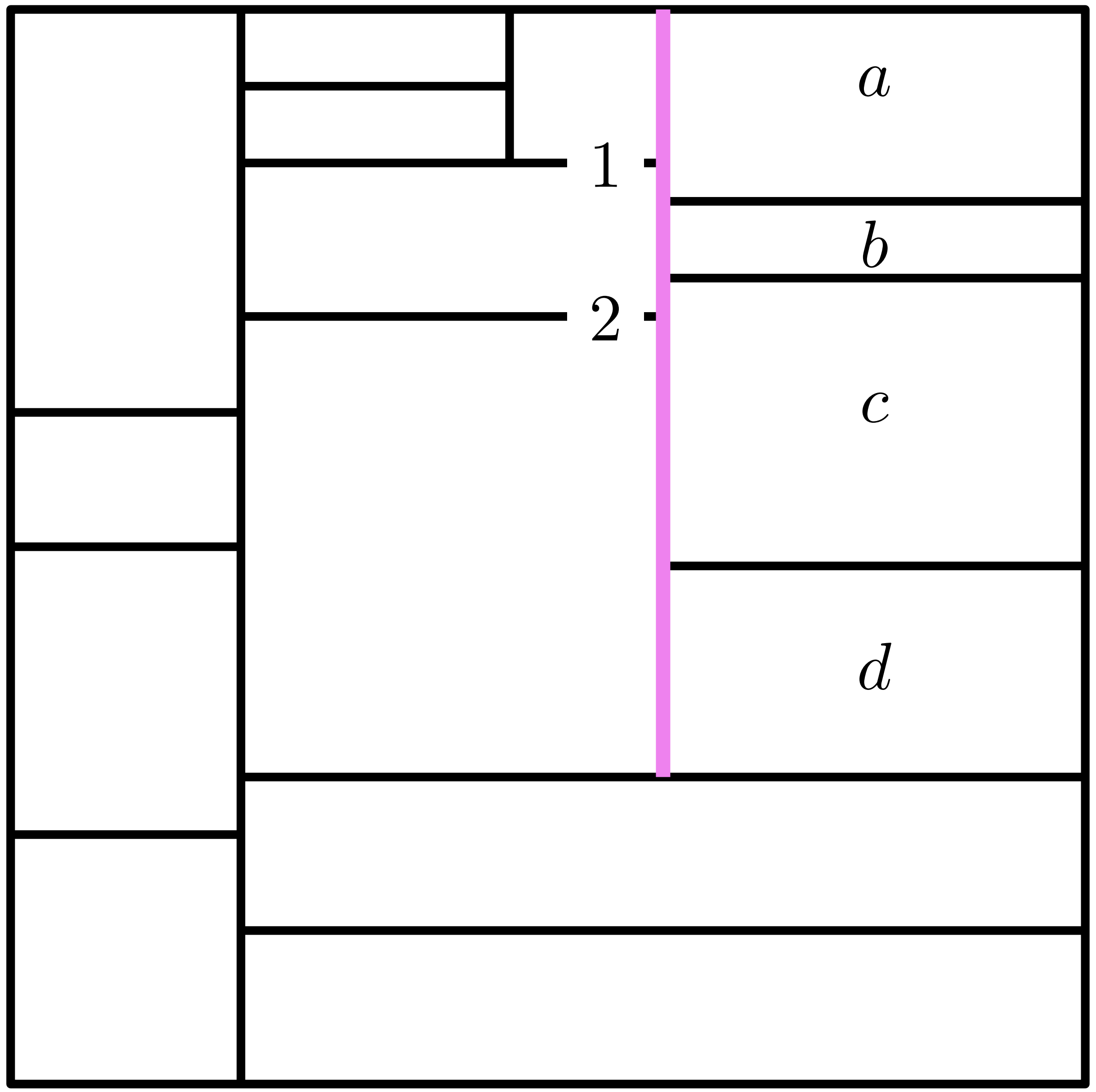

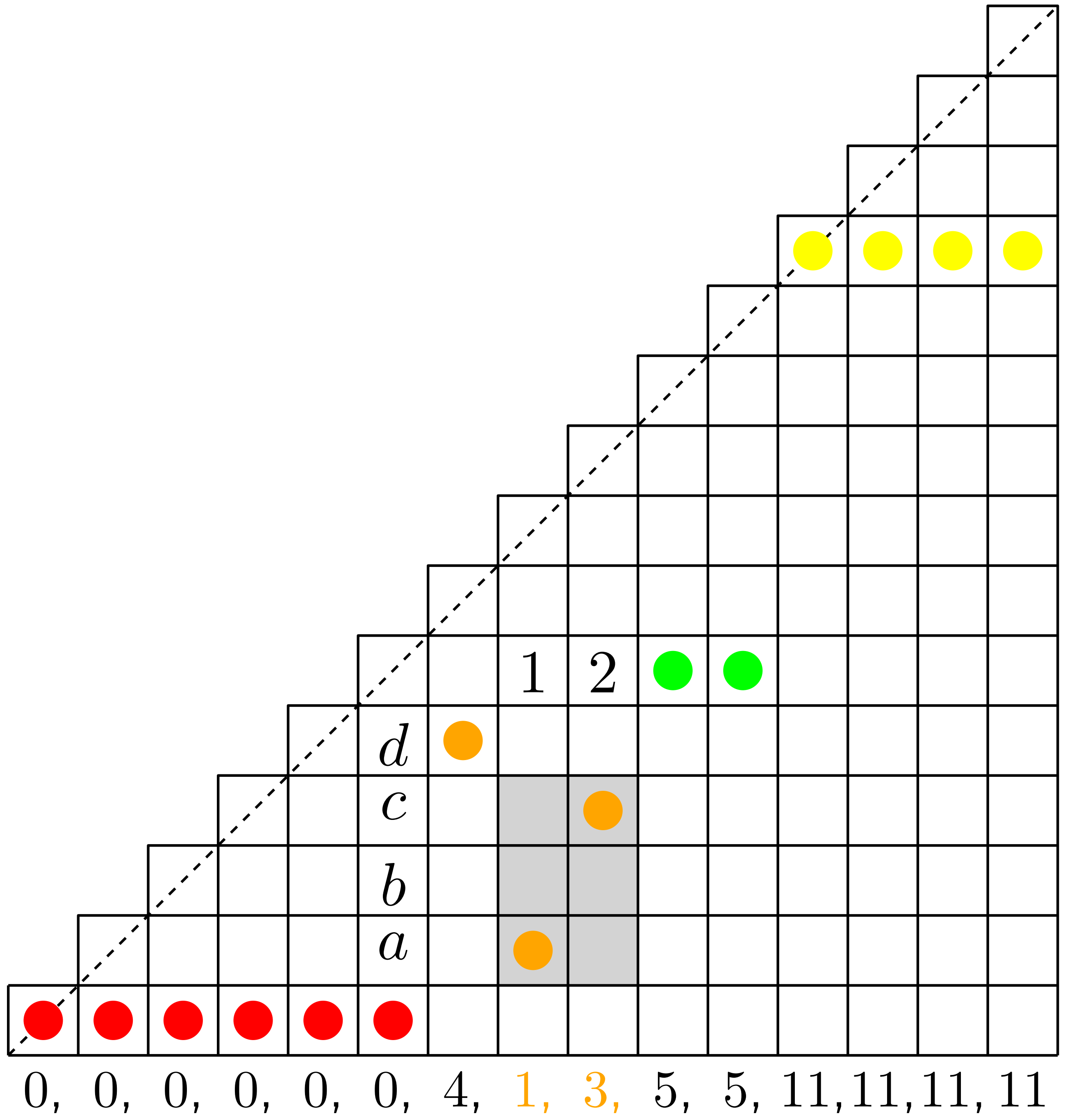

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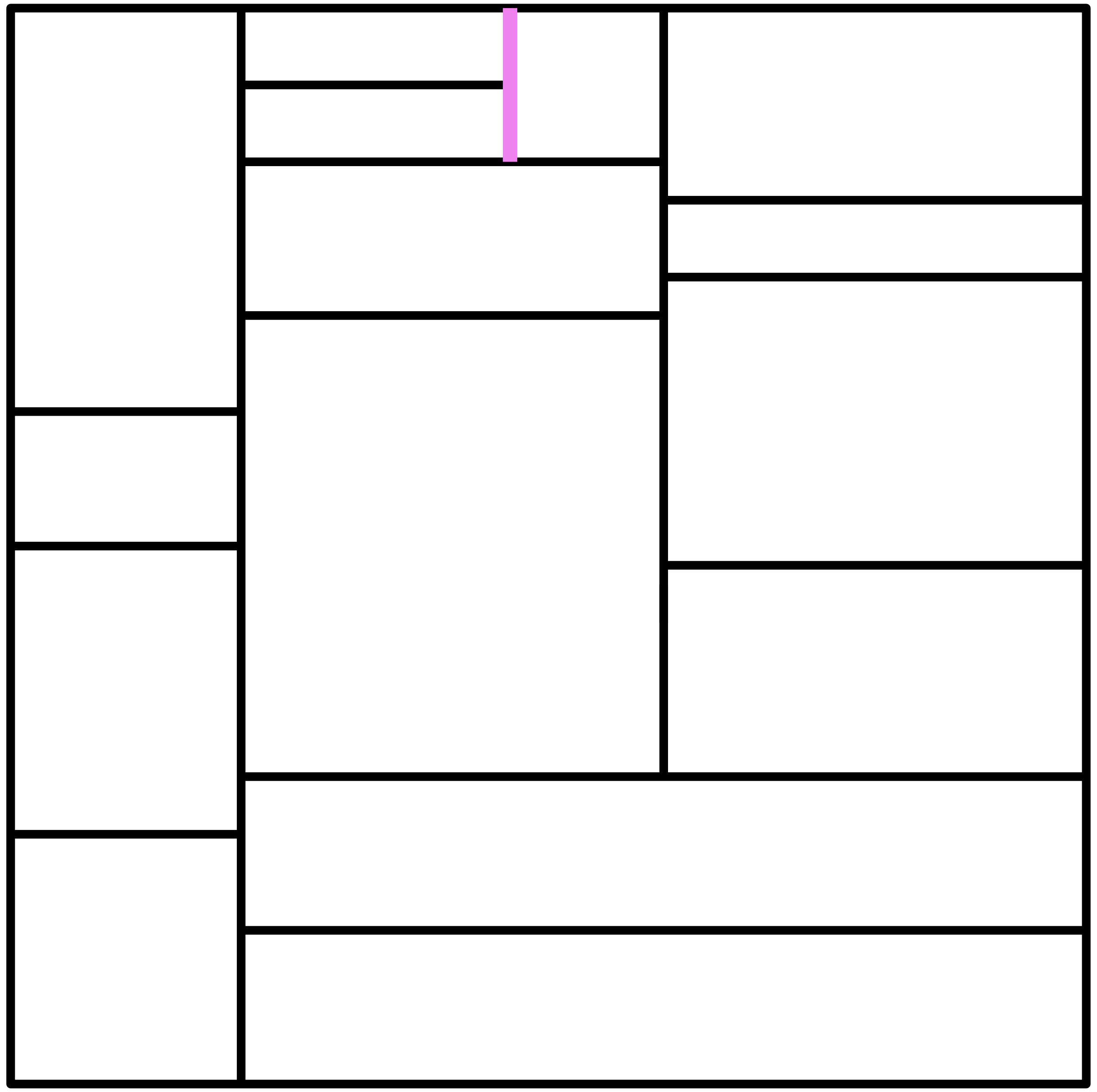

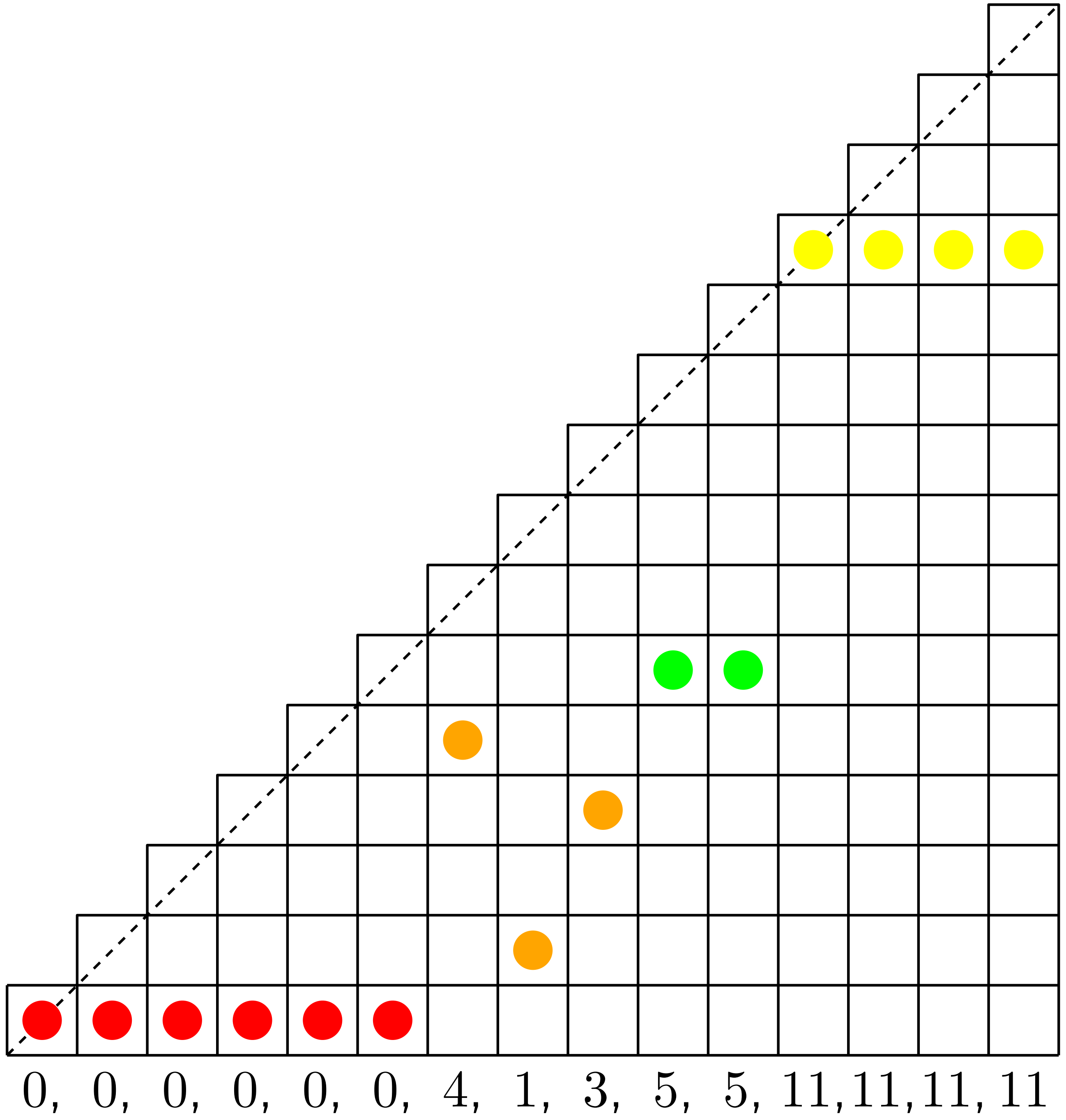

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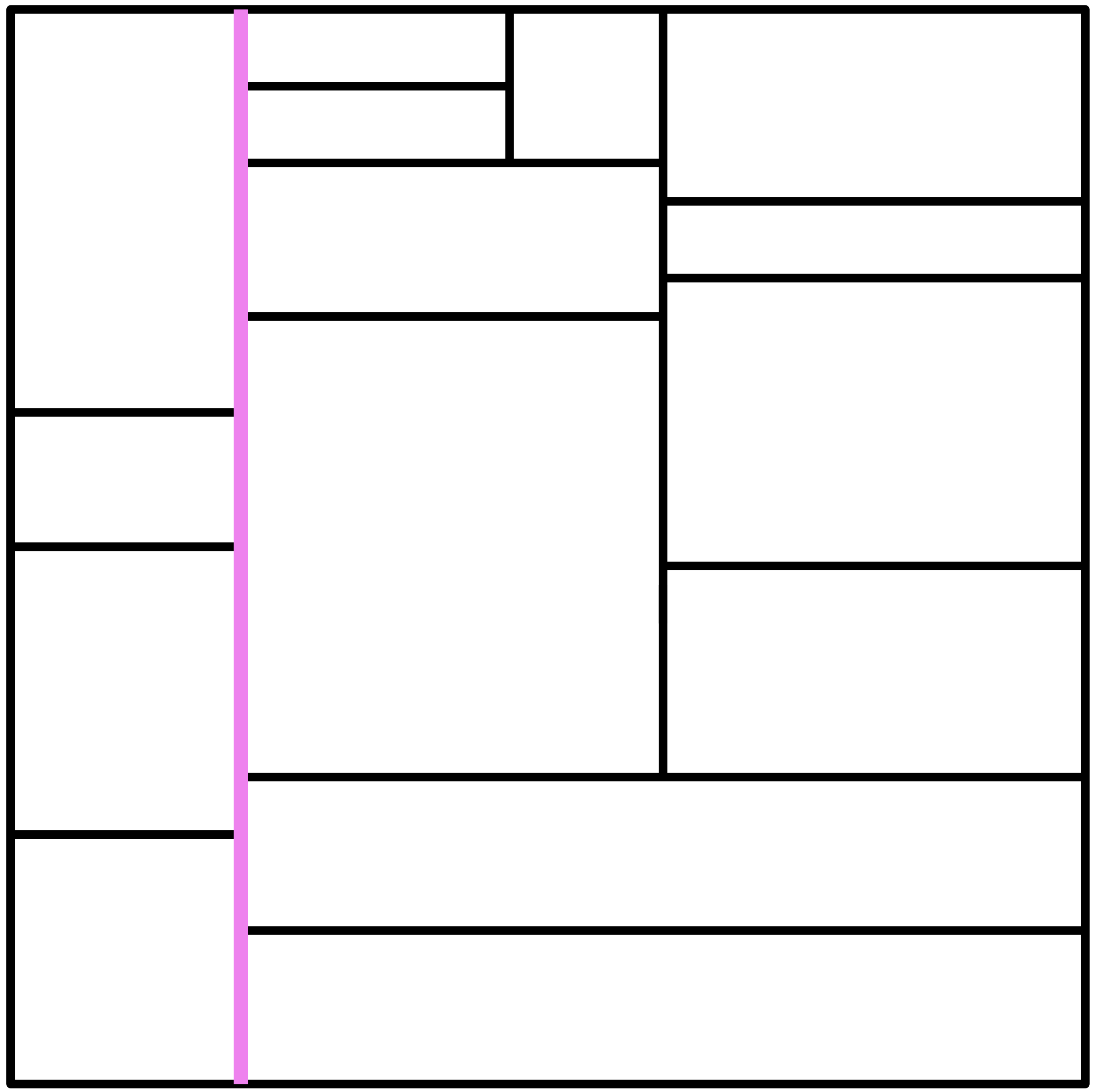

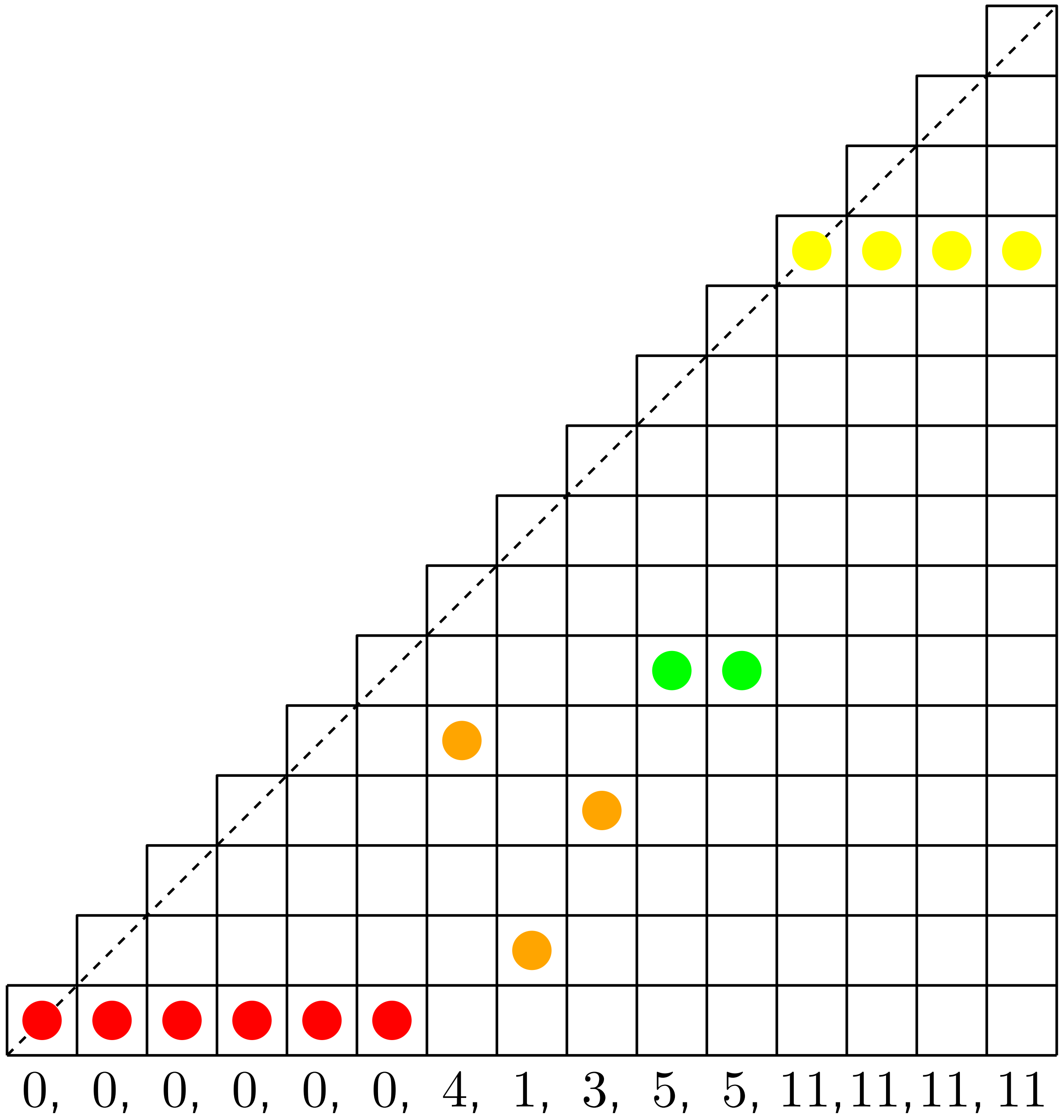

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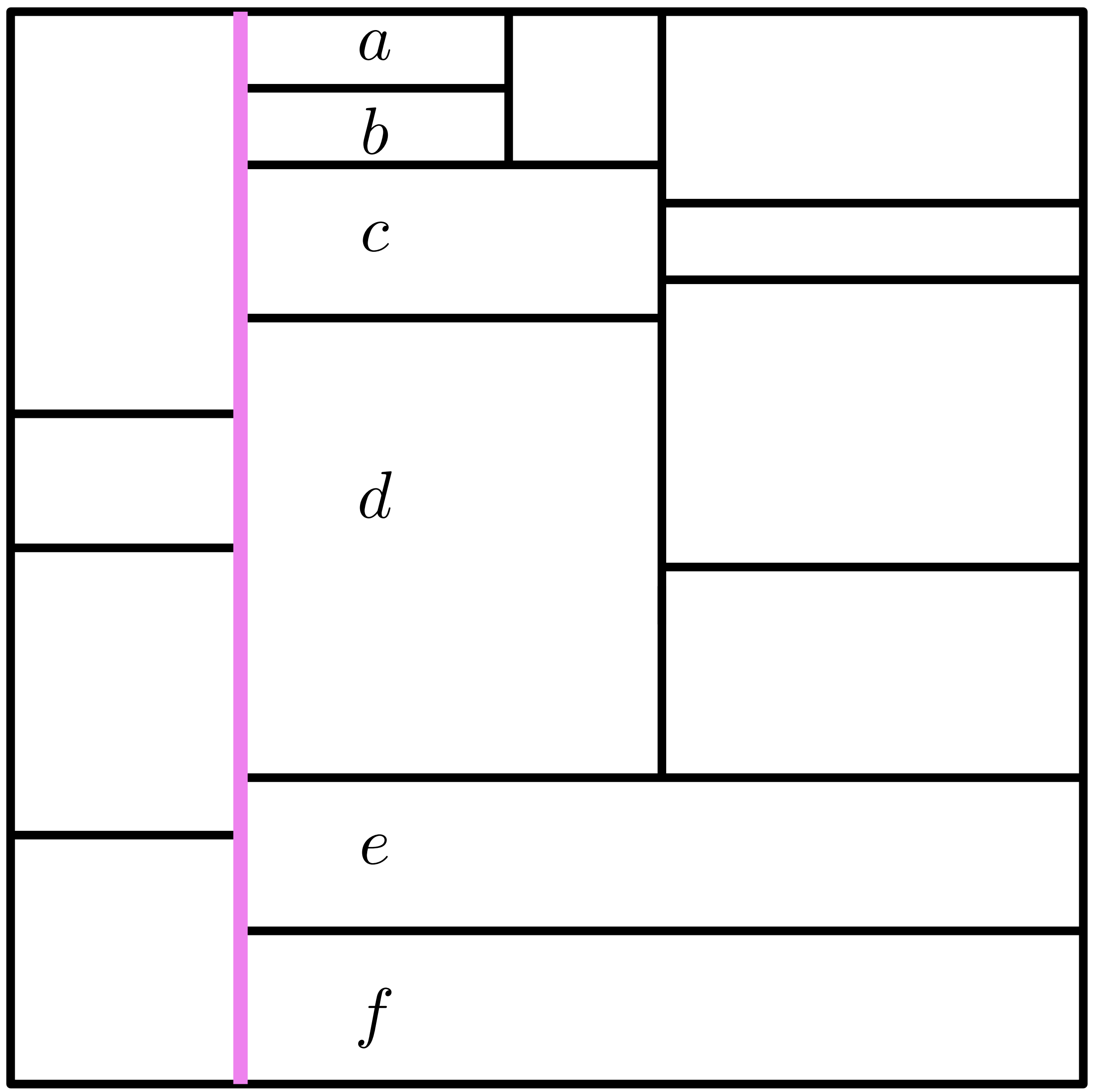

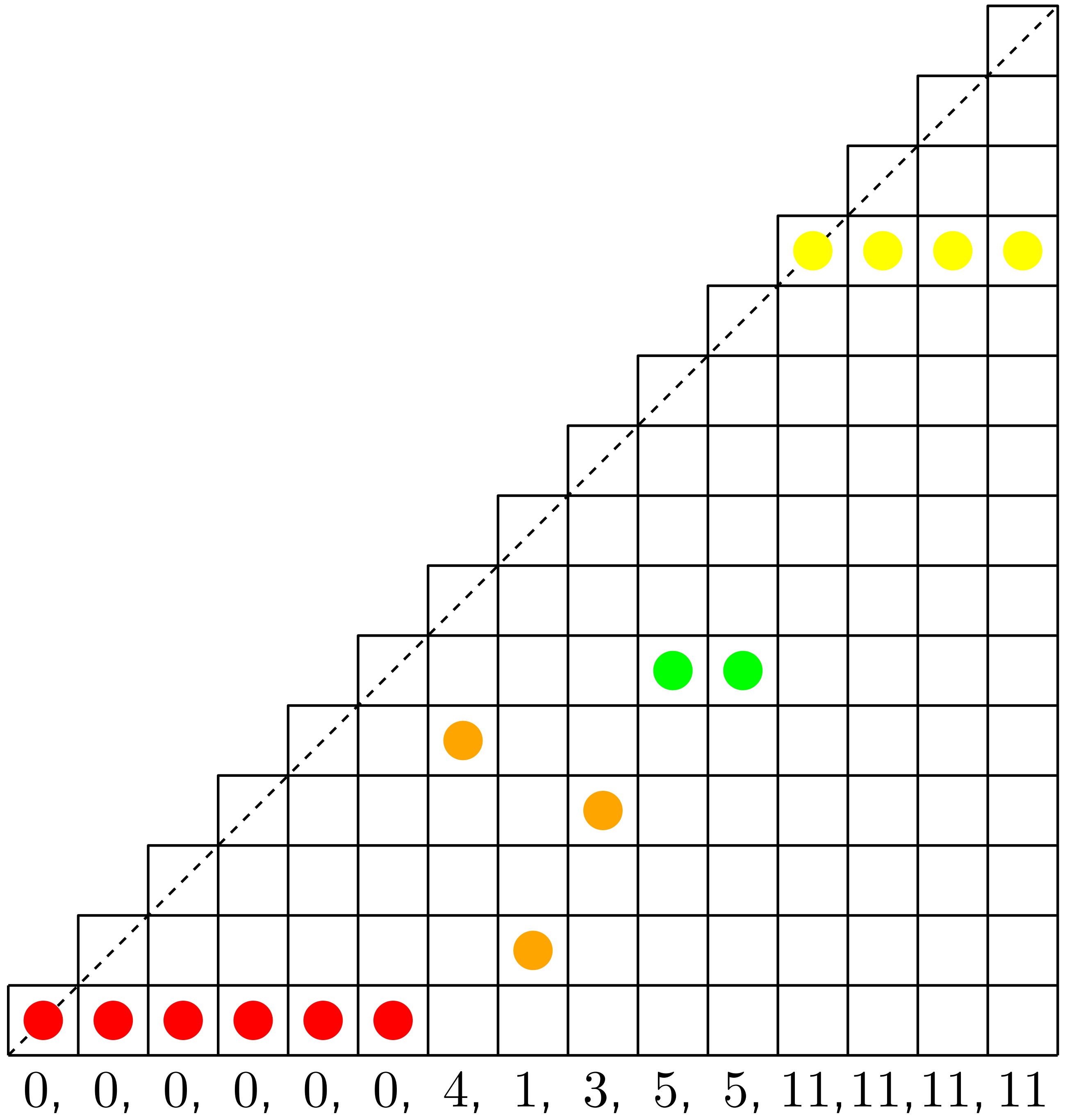

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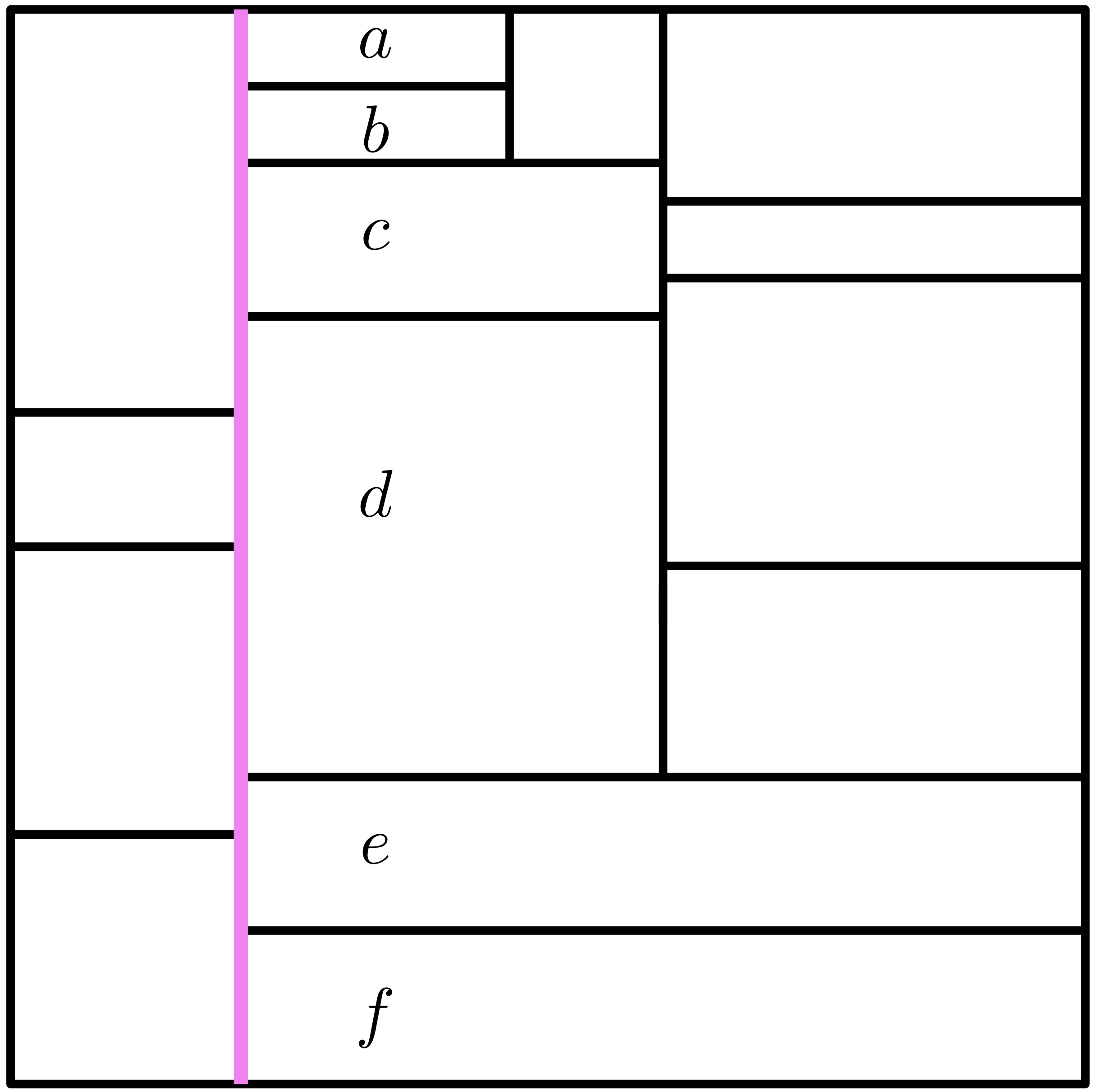

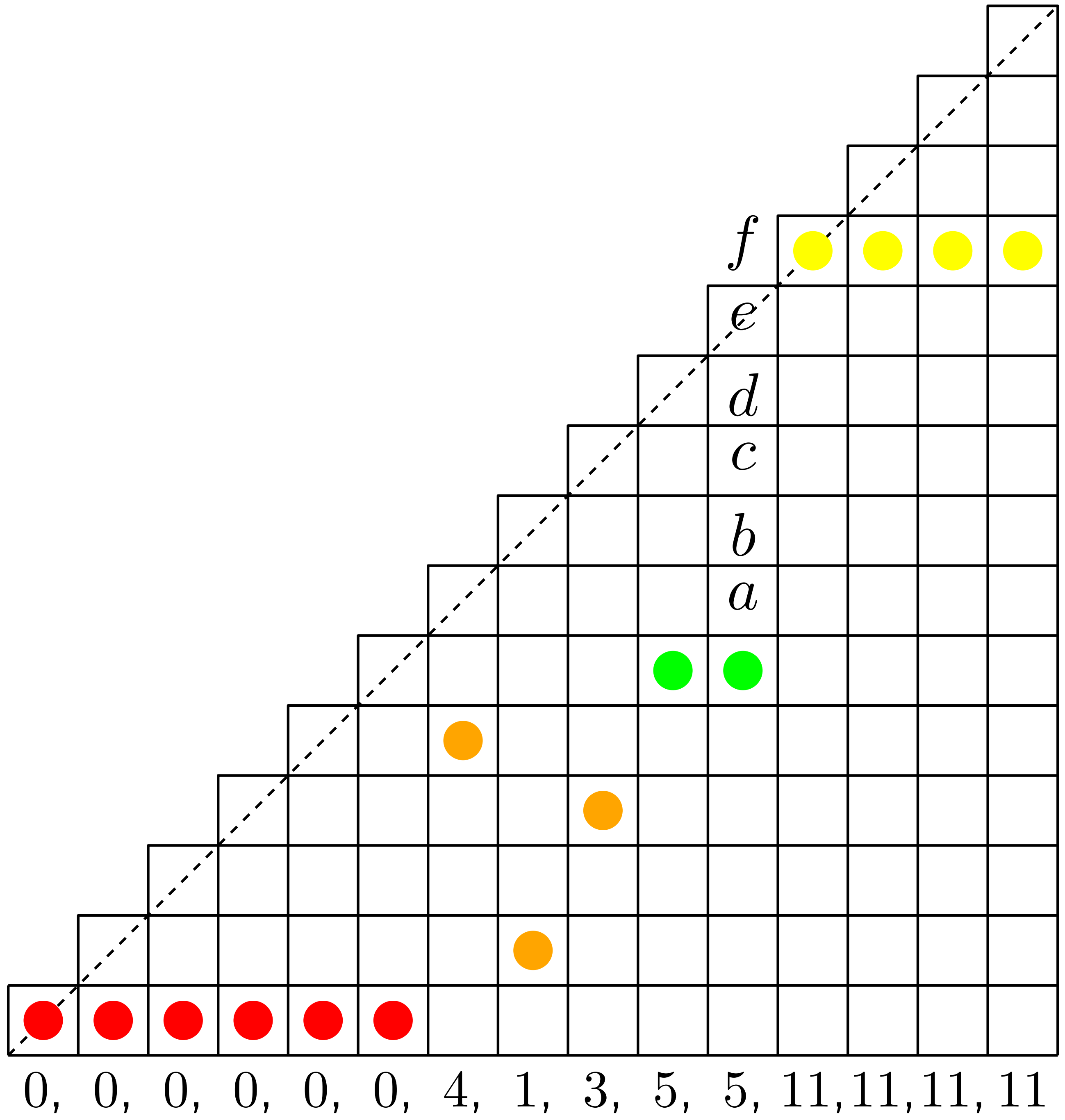

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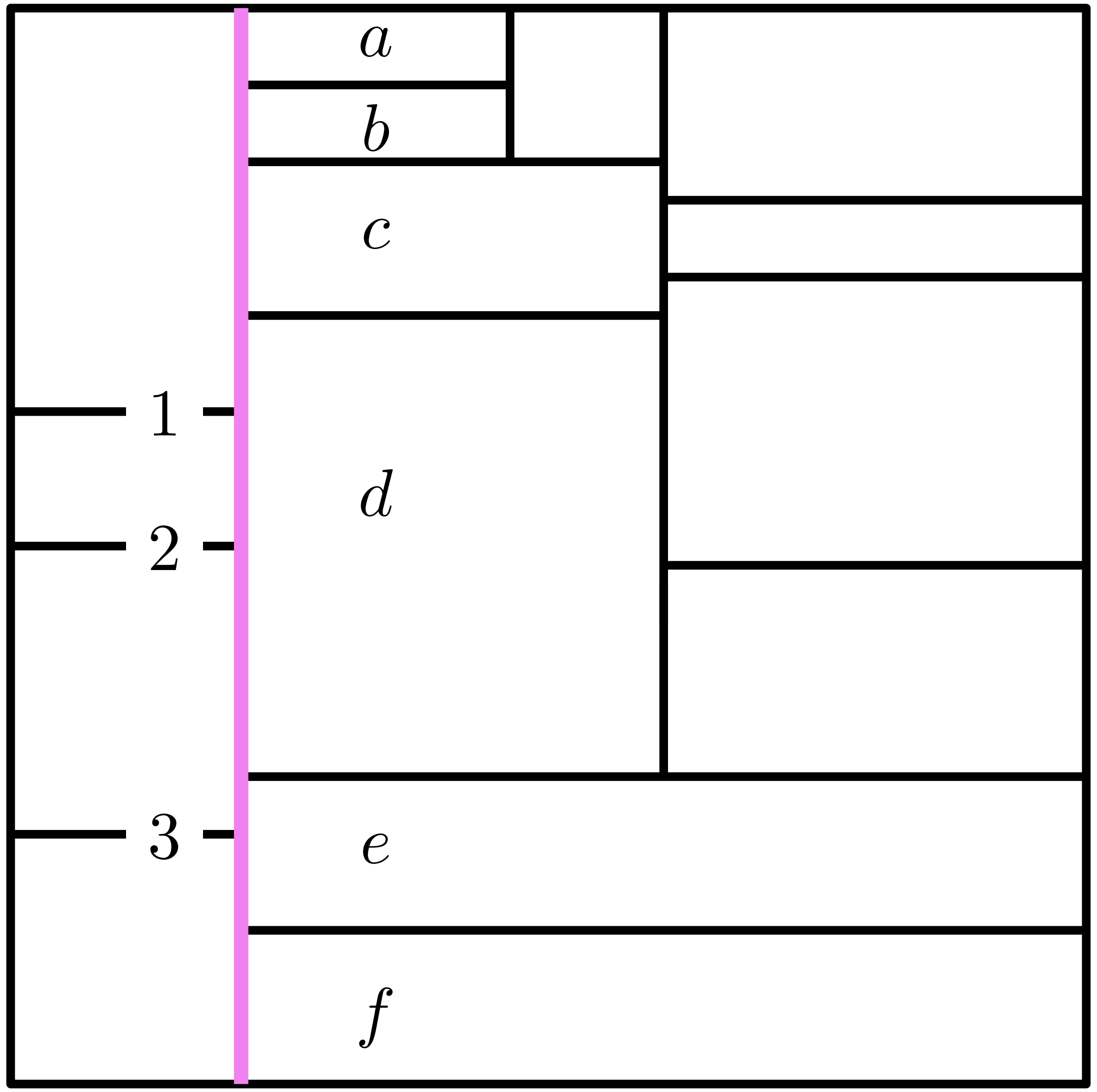

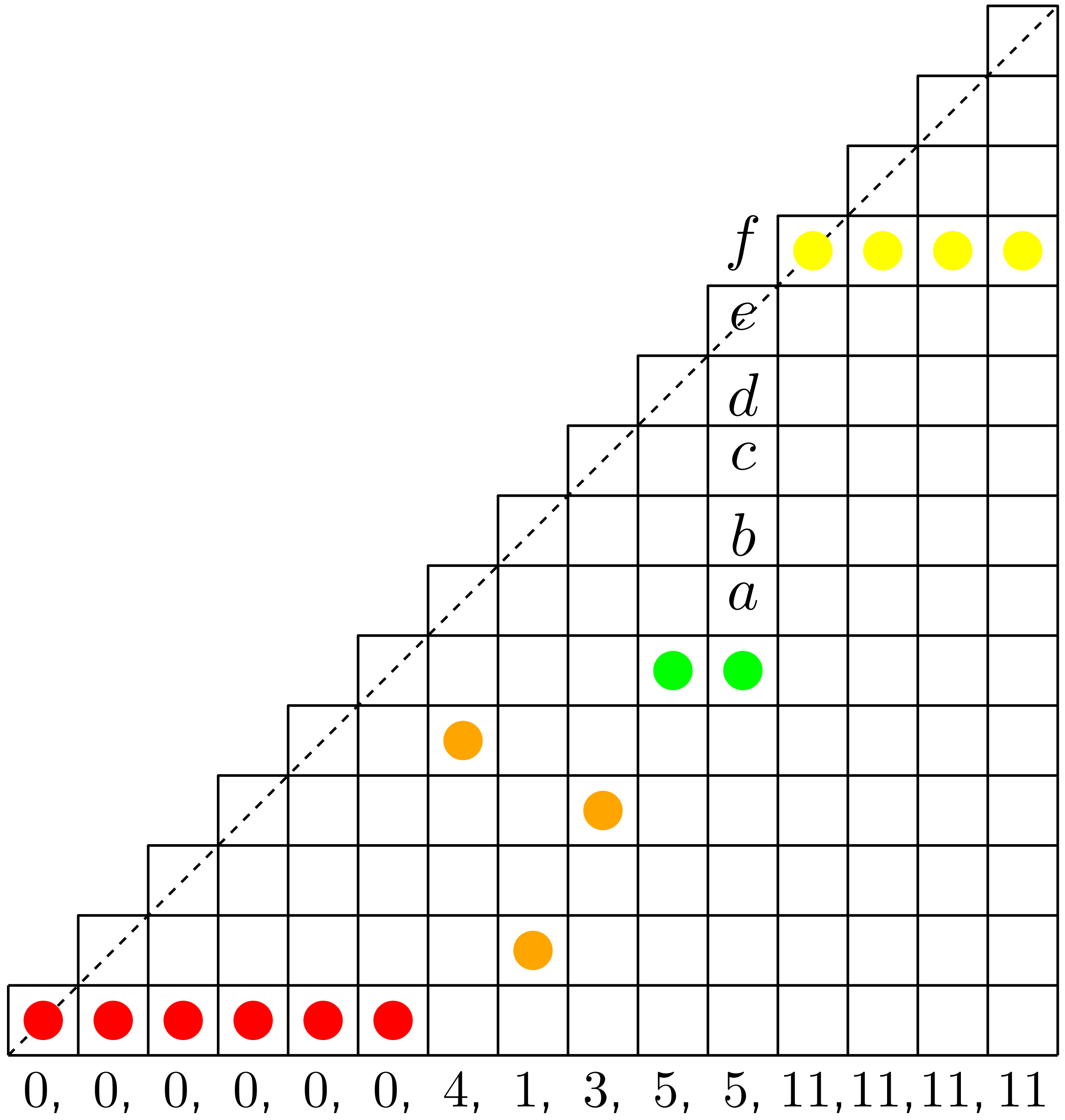

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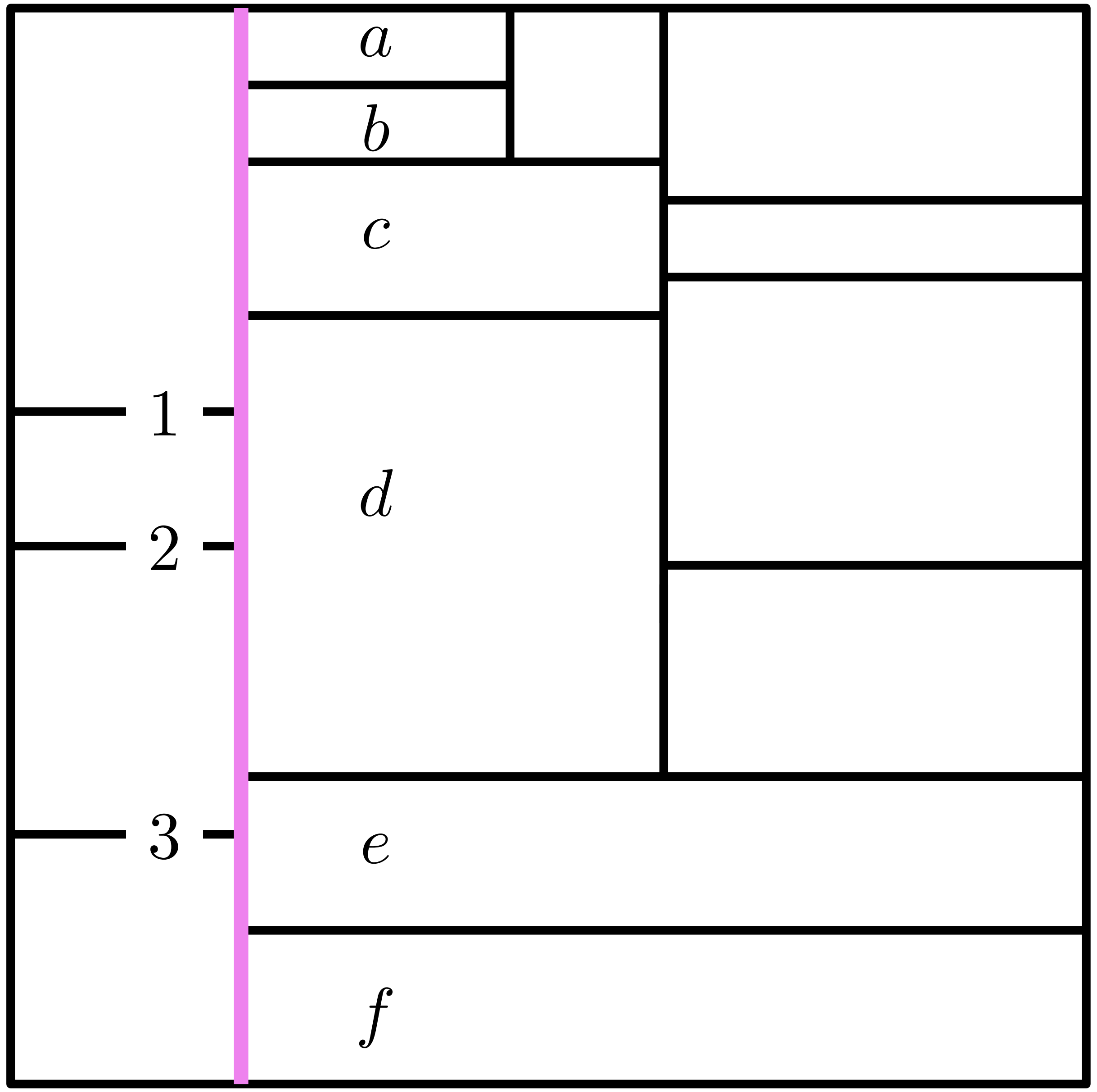

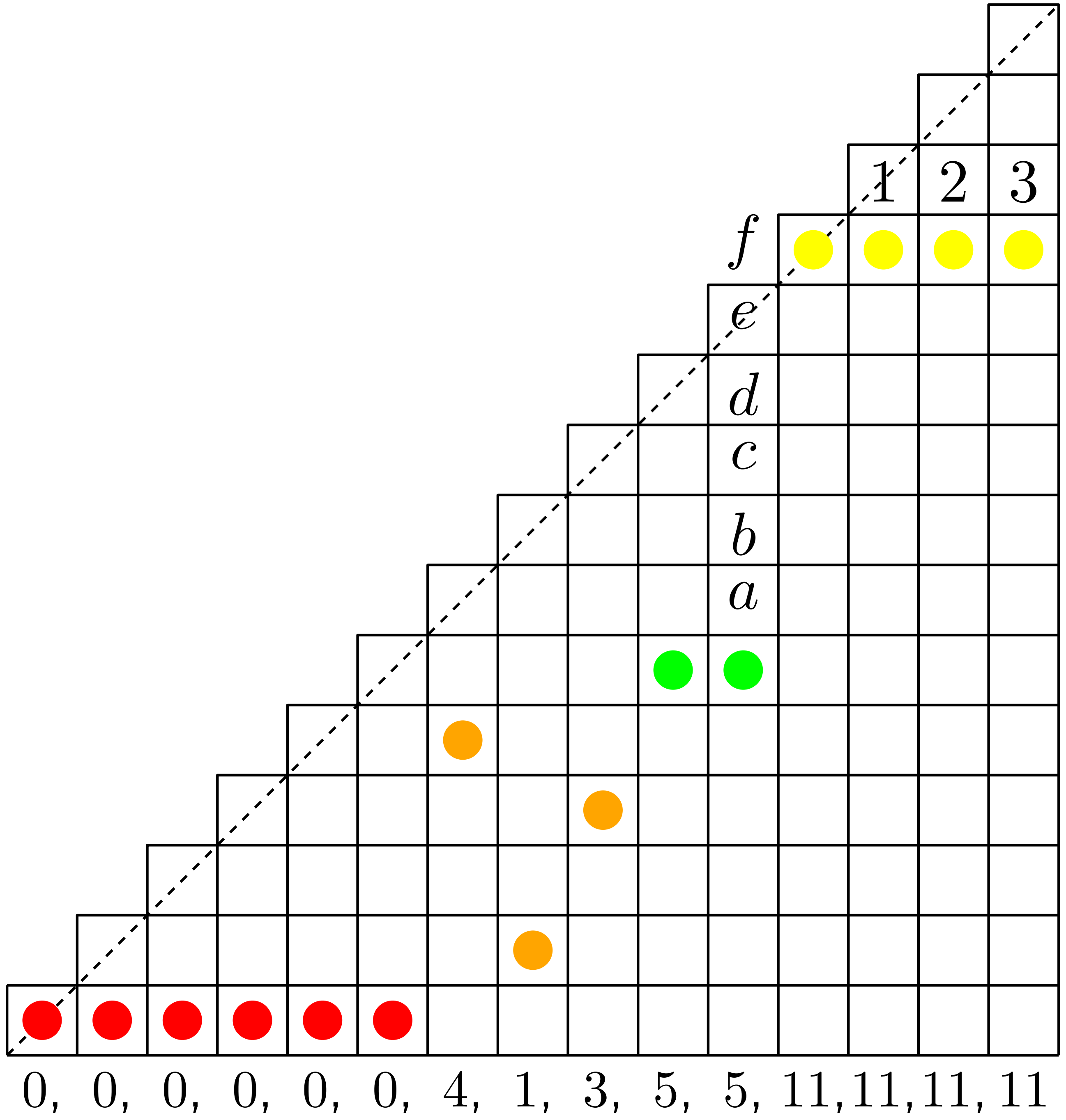

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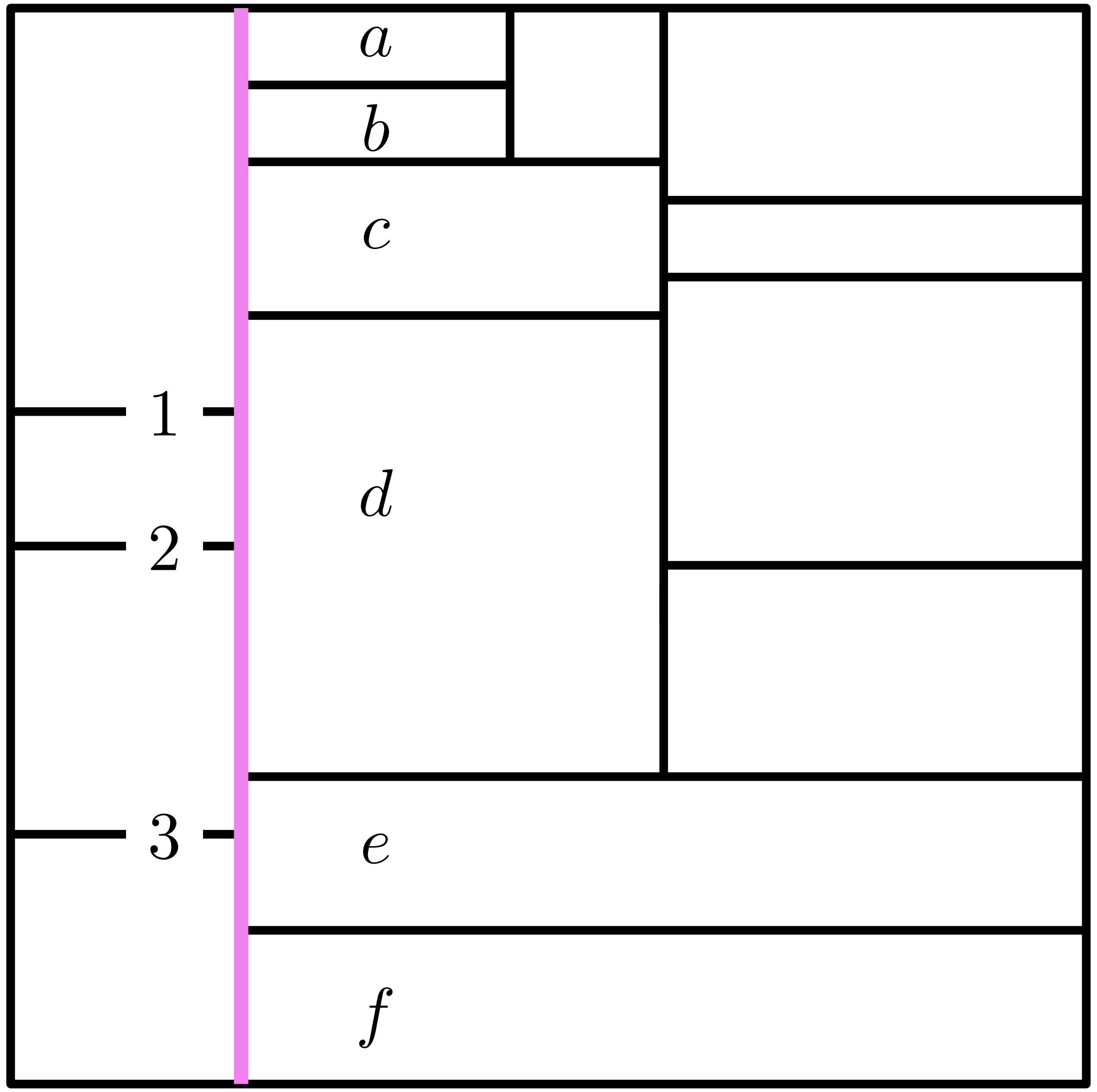

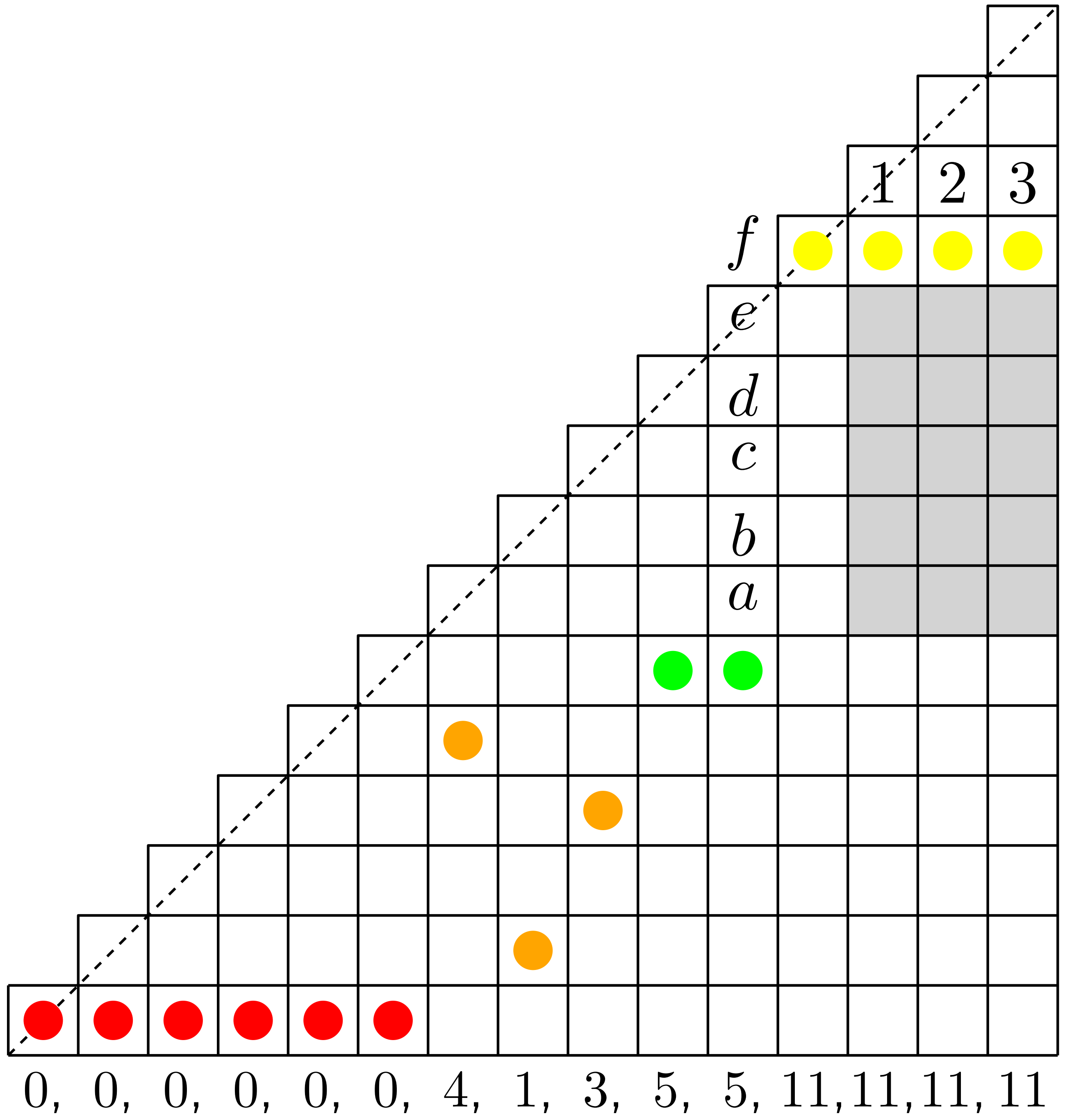

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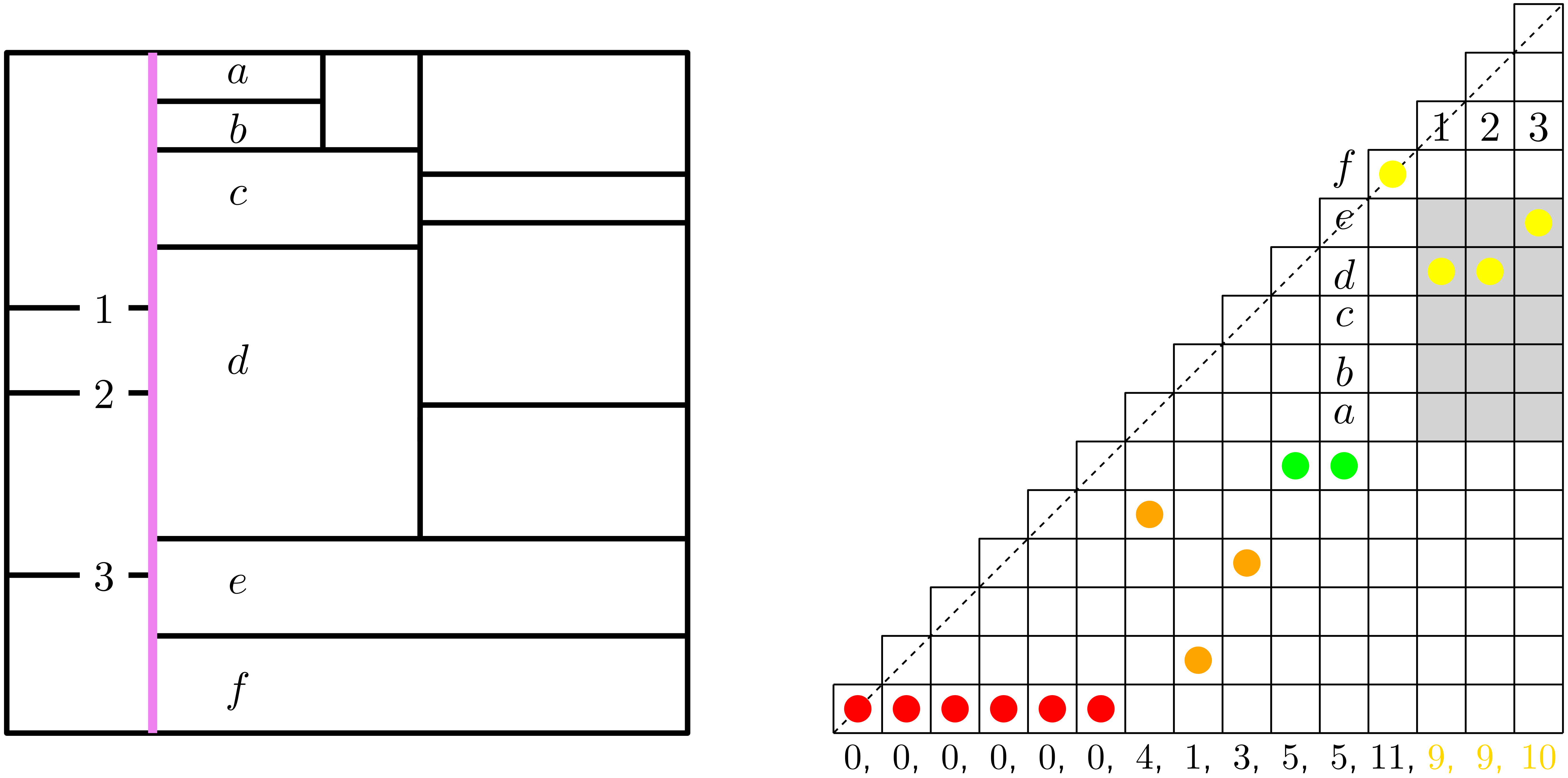

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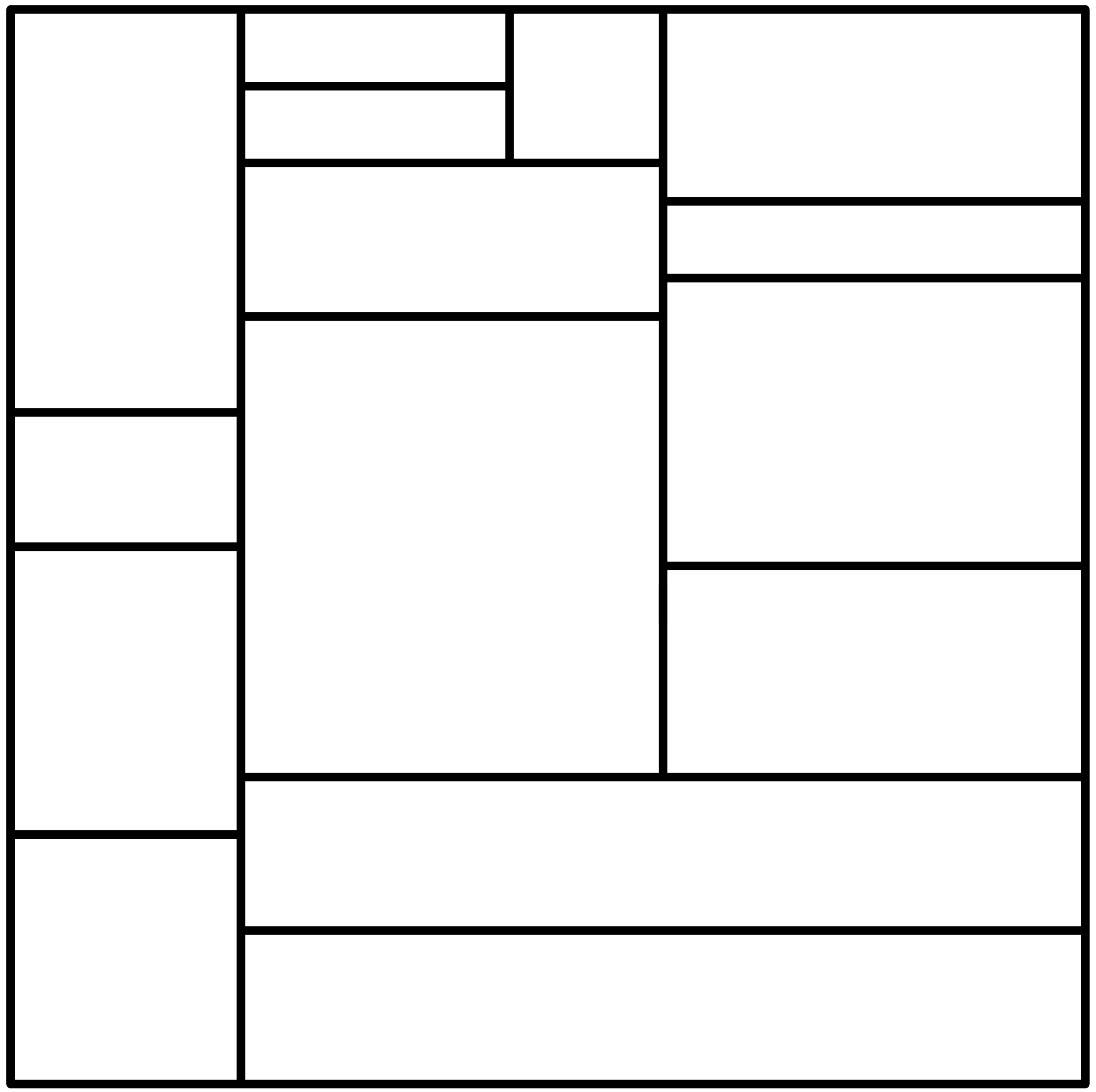

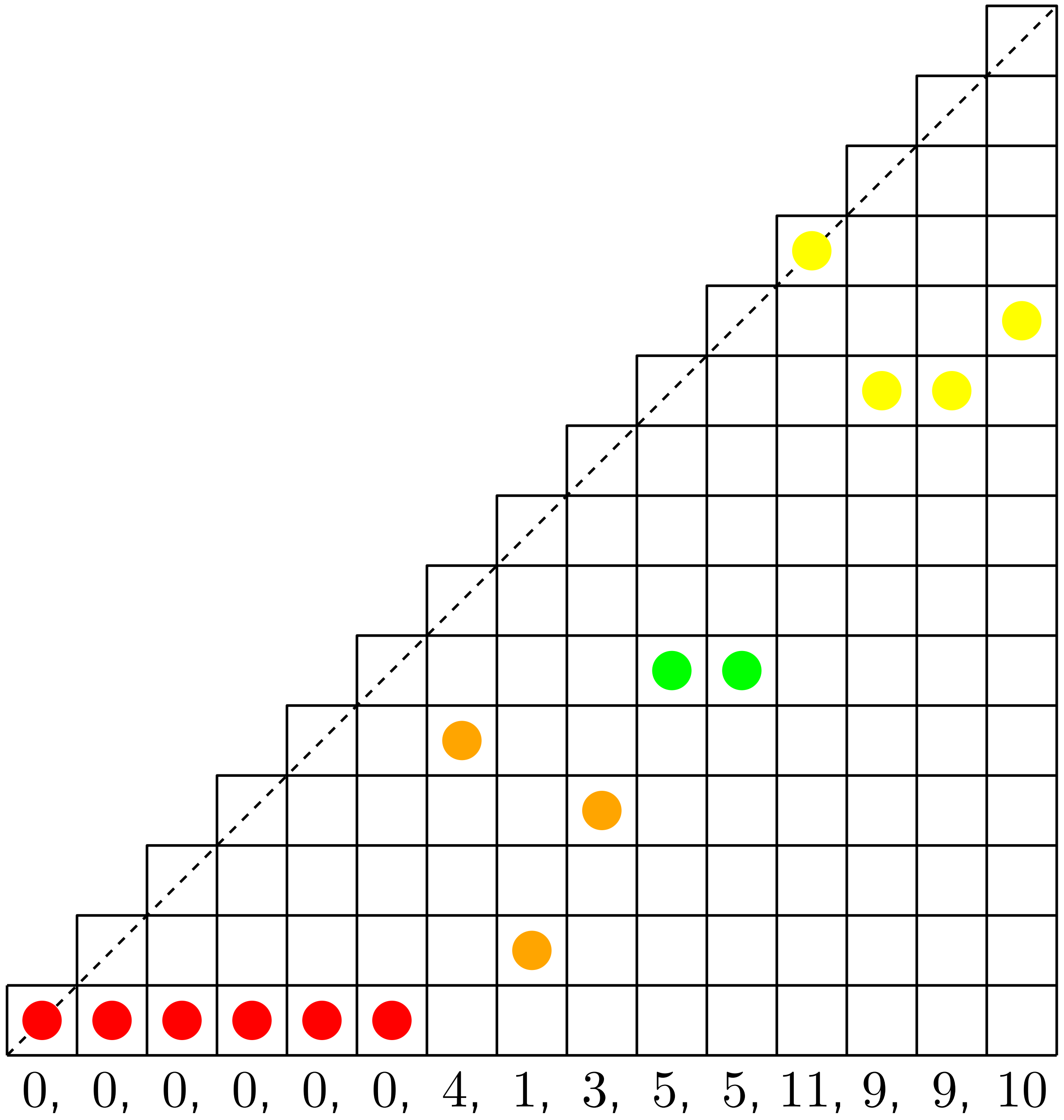

First geometric interpretation of sequence, sequence previously appeared in paper examining pattern avoidance in inversion sequences from Megan Martinez and Carla Savage (2018).

Proposition 3a: $|R_n^w(\vdash, \dashv)| = 2^{n-1}$

Proof: Enumerated by compositions of n.

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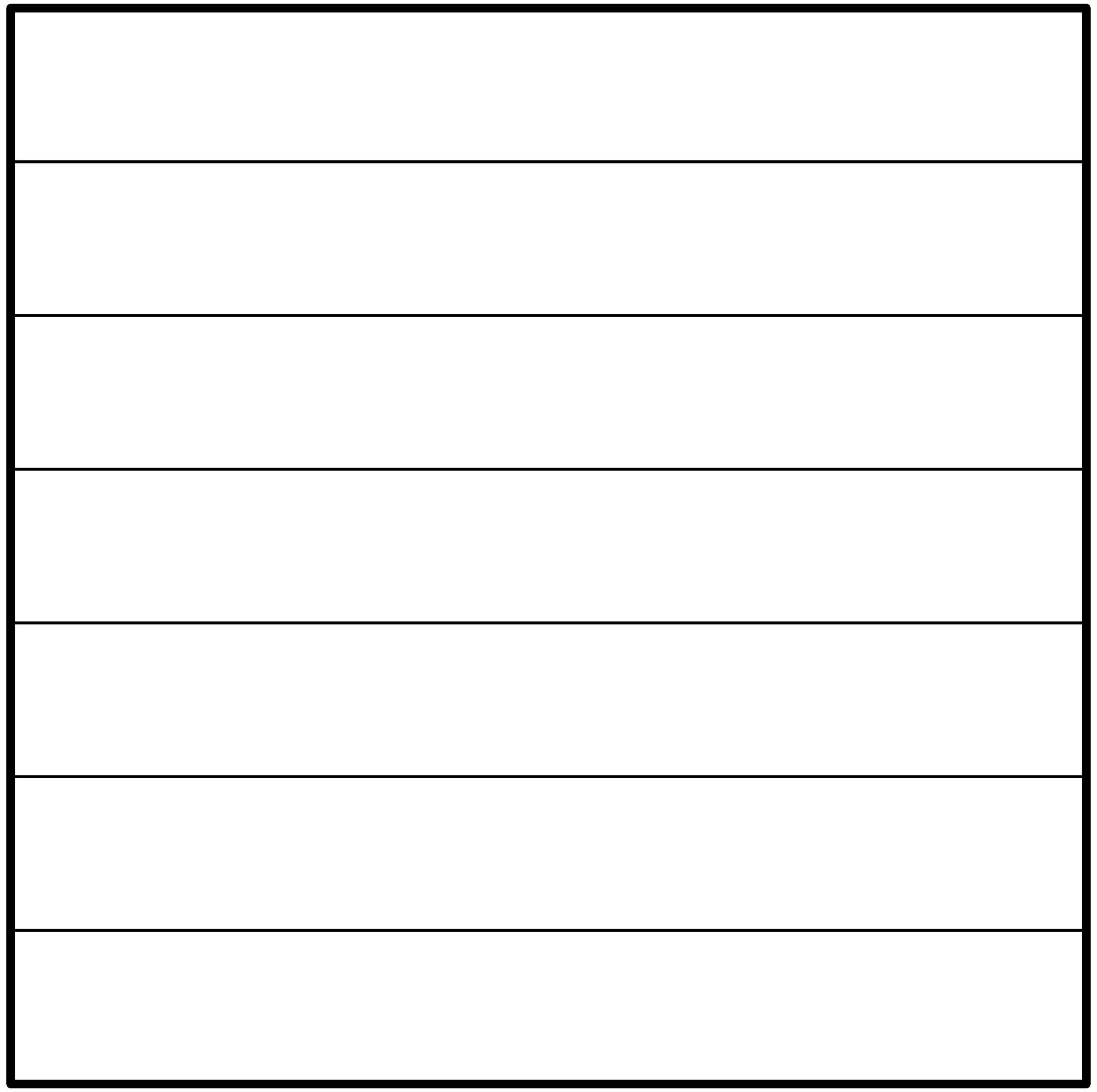

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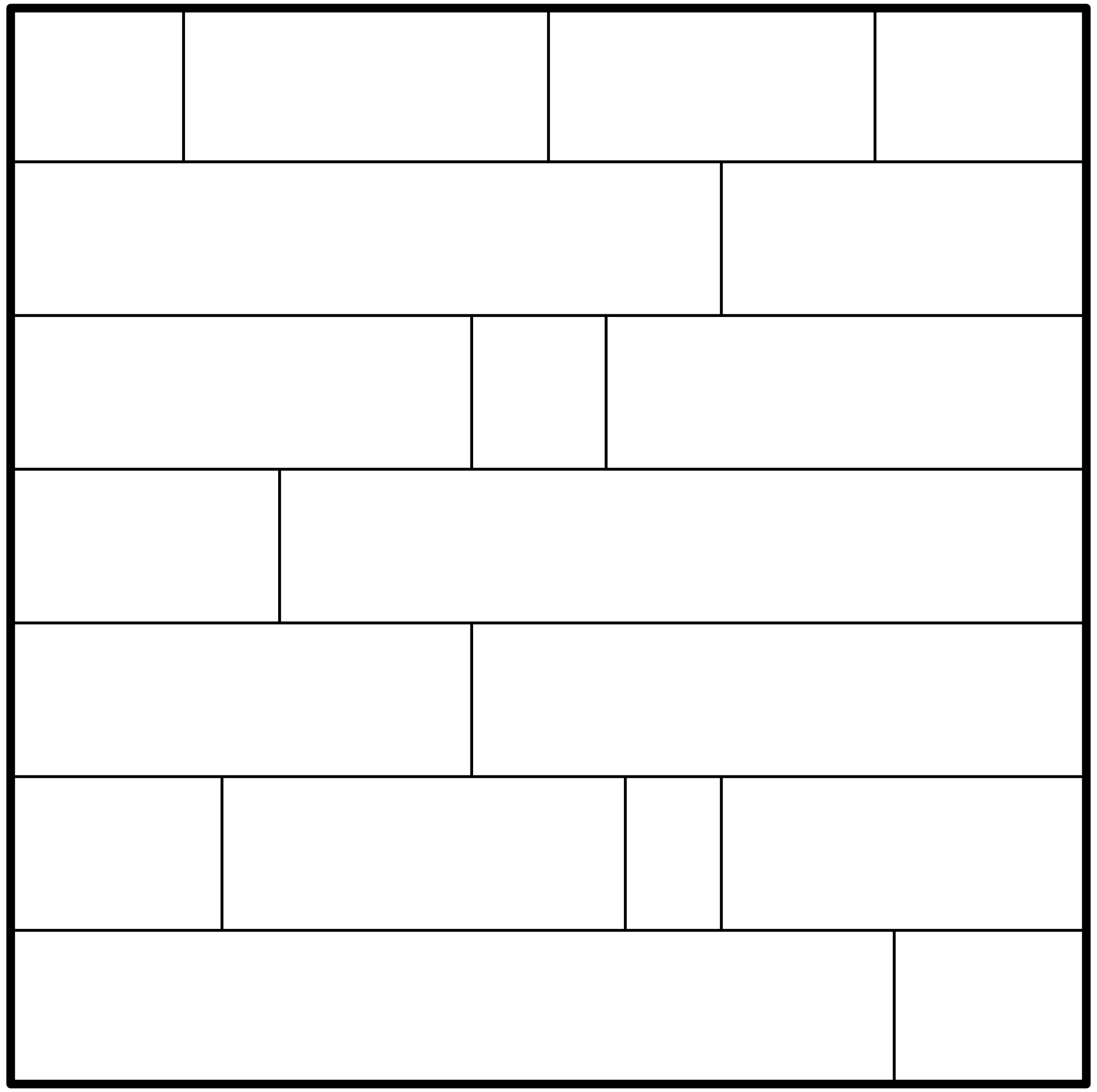

Proposition 3b (Asinowski and Jelínek): Enumerating $R_n^s(\vdash, \dashv)$, OEIS A287709

Proof: Bijection to rushed Dyck paths

A *rushed Dyck path* is one which attains its maximum height on the initial ascent.

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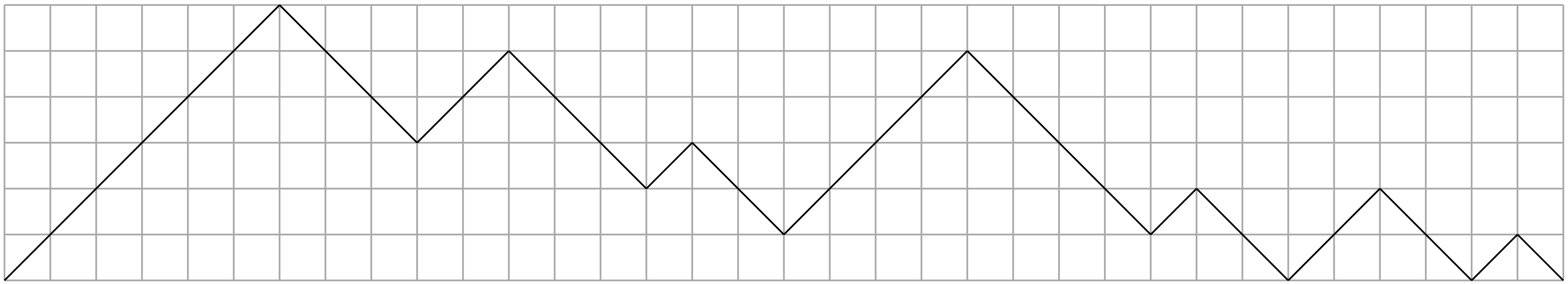

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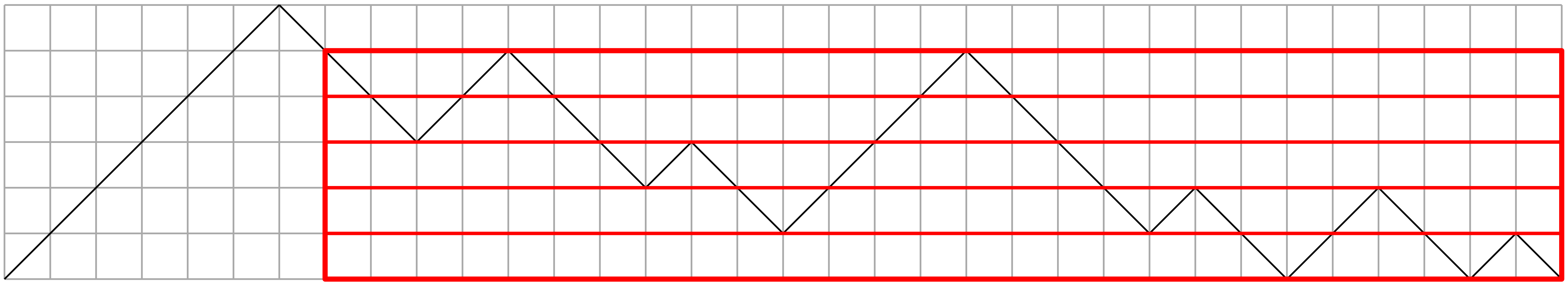

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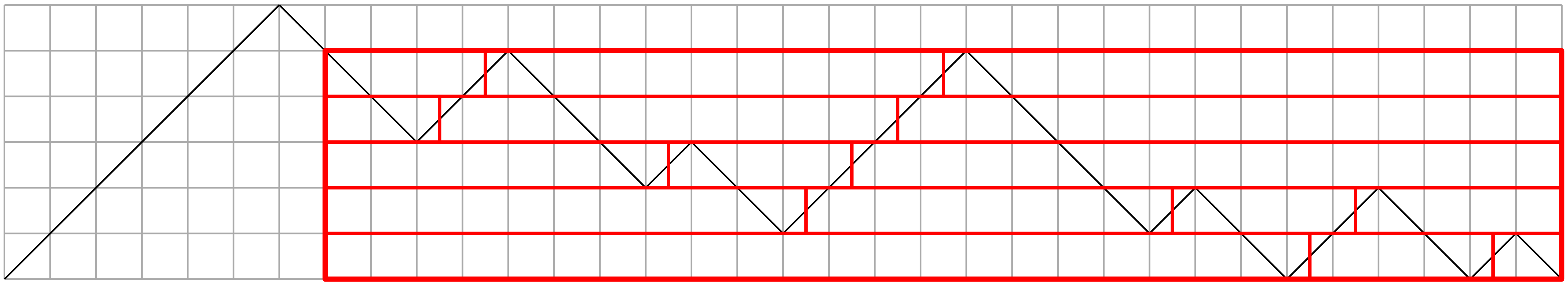

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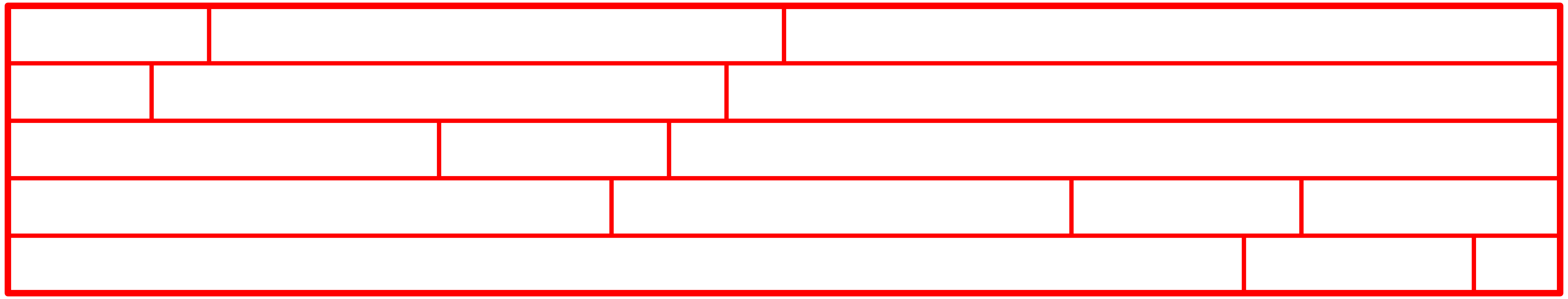

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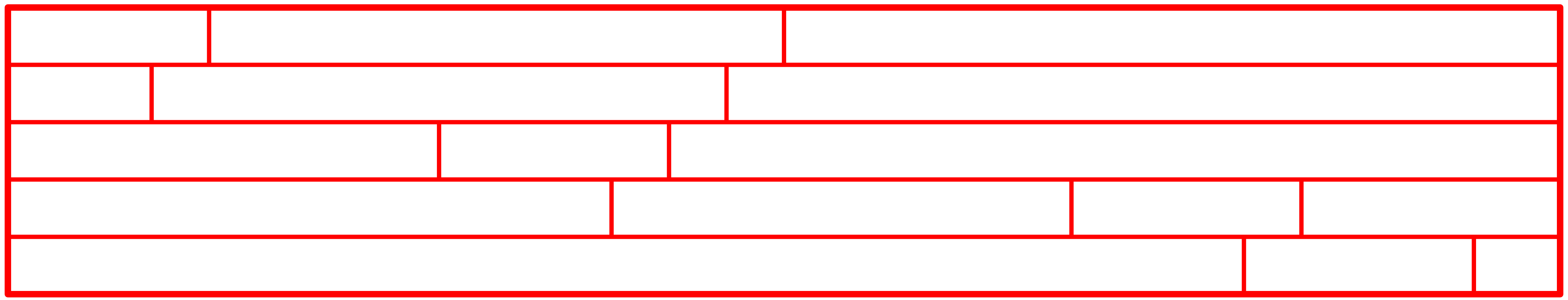
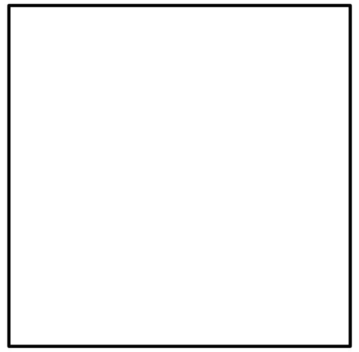
Asymptotics recently proven in a pre-print from Axel Bacher



Proposition 4: $|R_n(\top, \vdash)| = 2^{n-1}$

Proof: Construction of rectangulation



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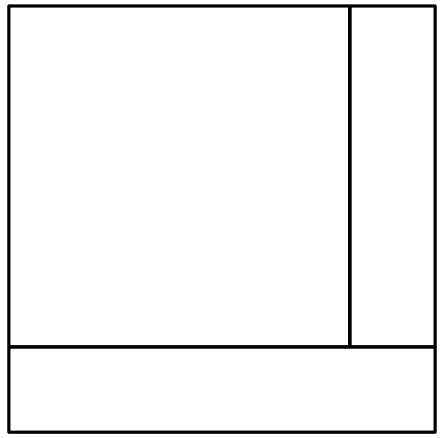
Proof: Construction of rectangulation



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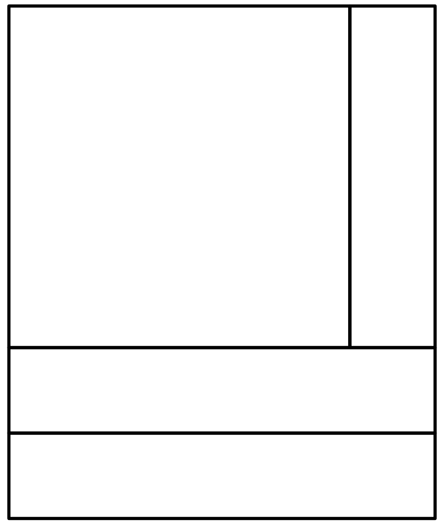
Proof: Construction of rectangulation



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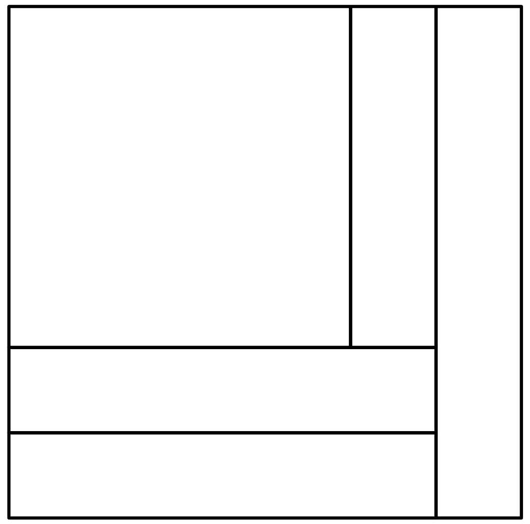

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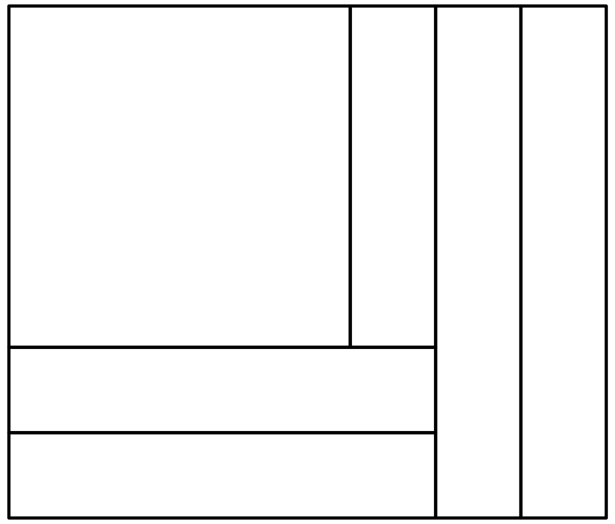

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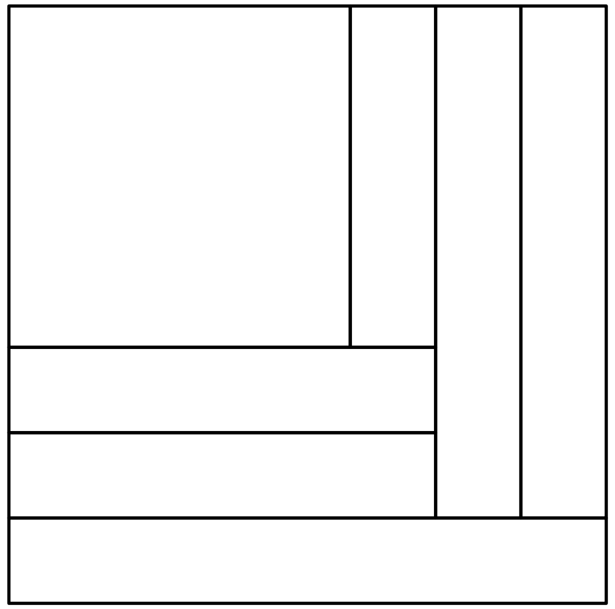

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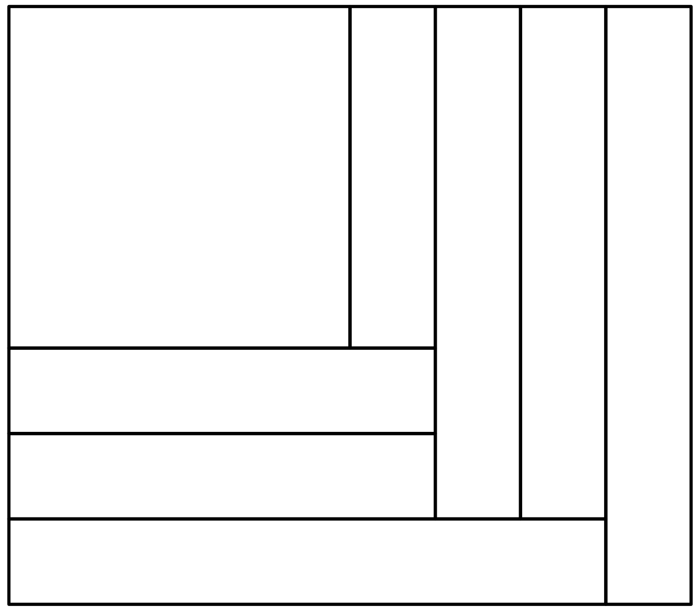

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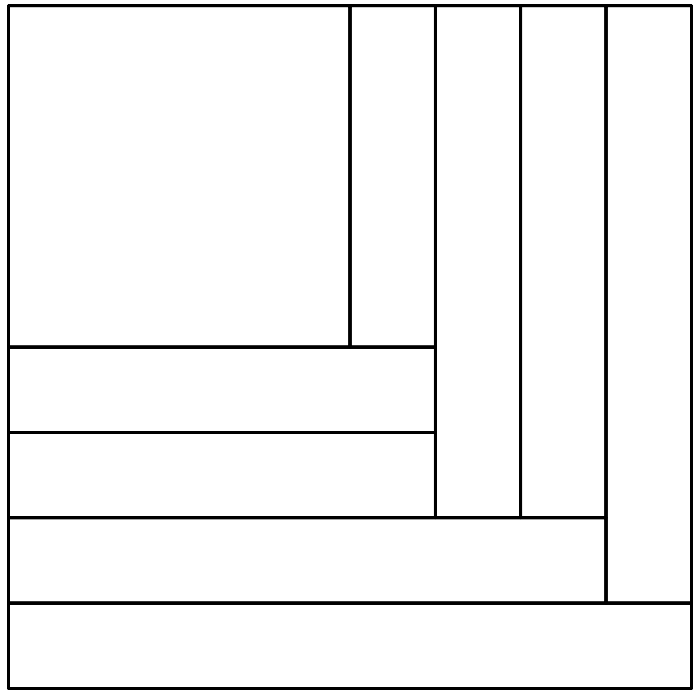
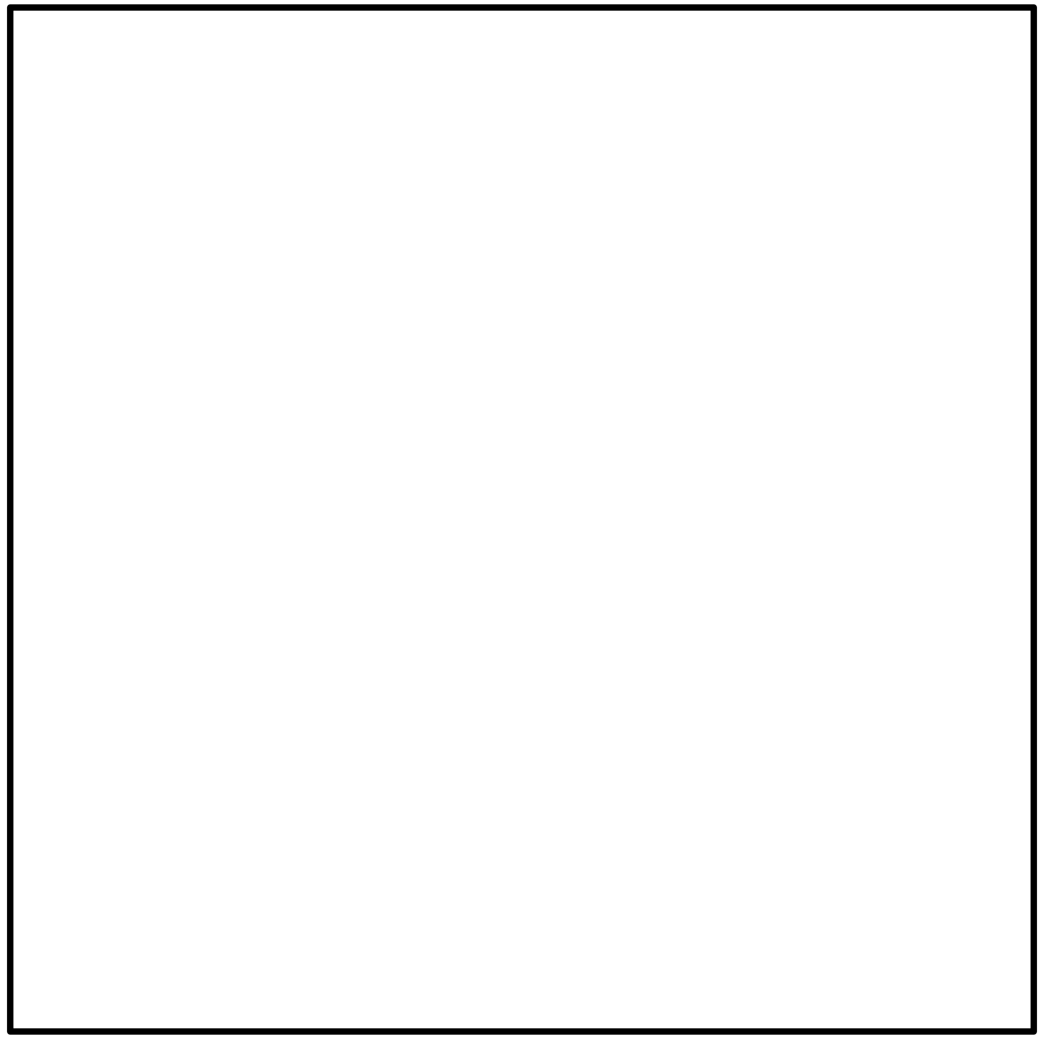
Proof: Construction of rectangulation



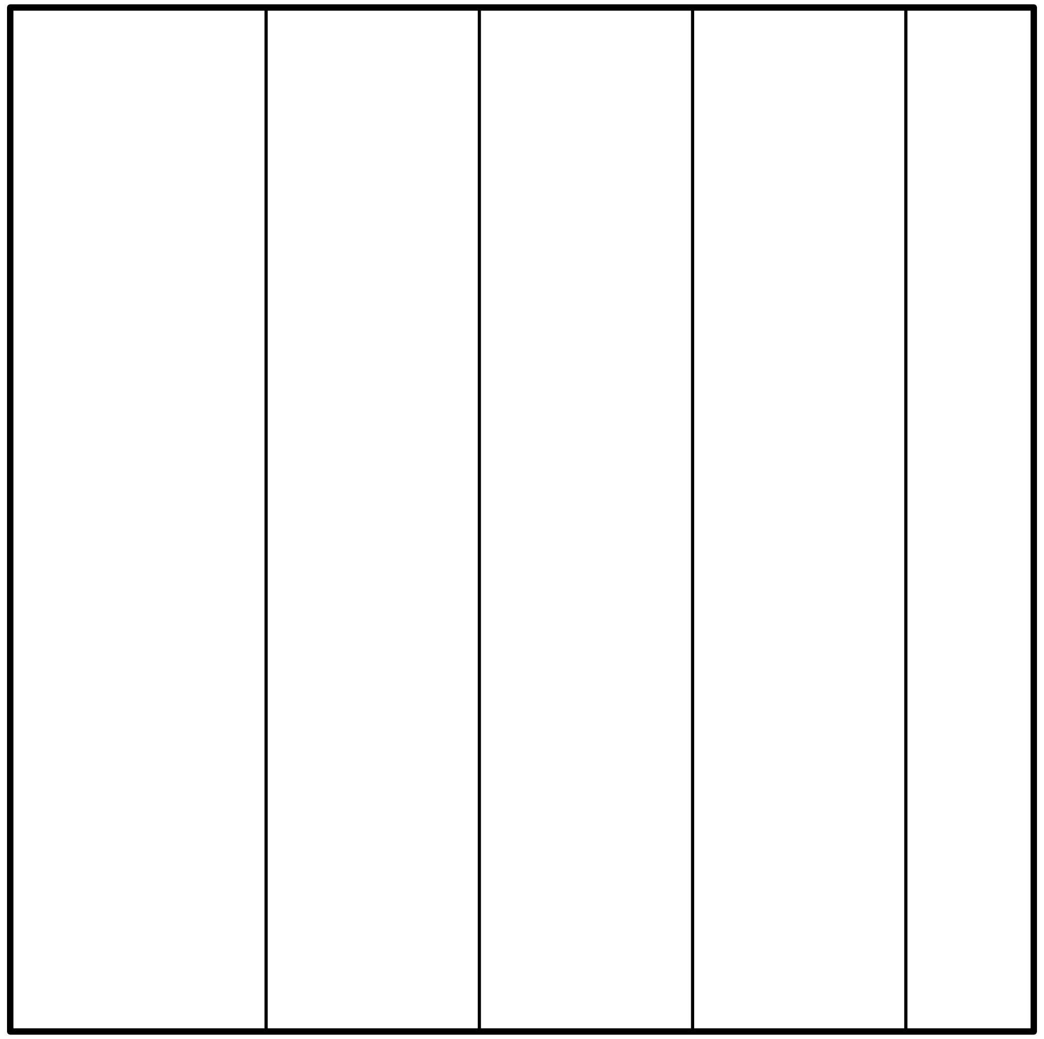
Observation 5: $|R_n(\top,\bot,\vdash)| = n$ and $|R_n(\top,\bot,\vdash,\dashv) = 2$

Proofs: Construction of rectangulations



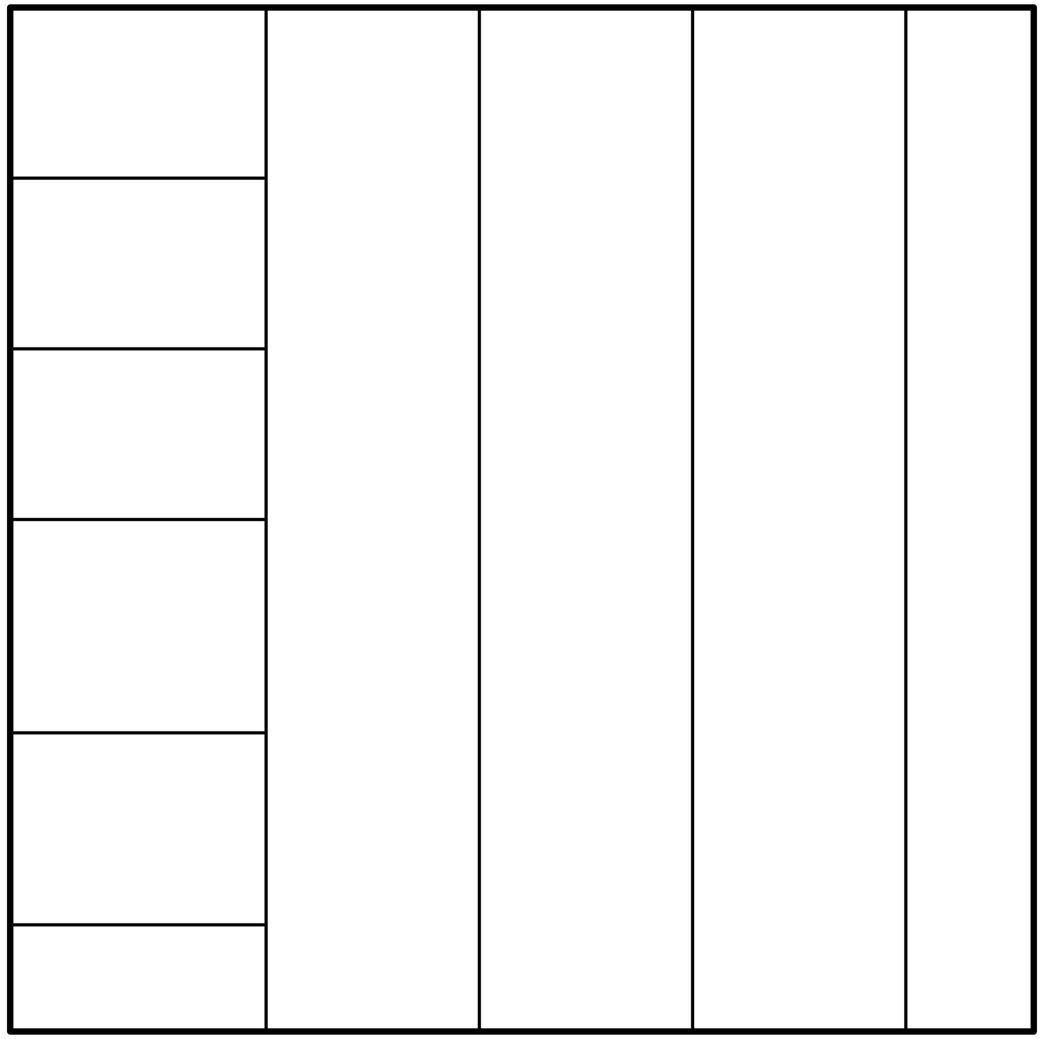
Observation 5: $|R_n(\top,\bot,\vdash)| = n$ and $|R_n(\top,\bot,\vdash,\dashv) = 2$

Proofs: Construction of rectangulations



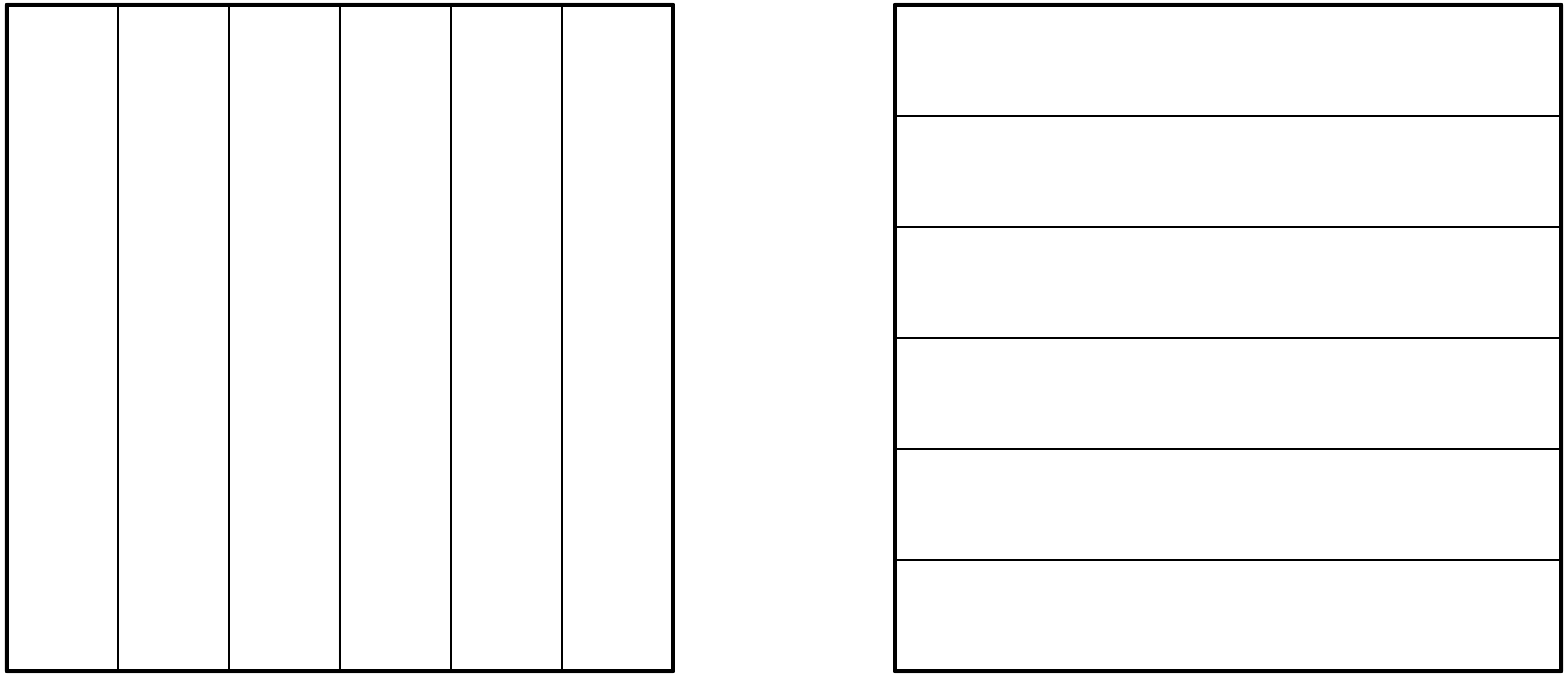
Observation 5: $|R_n(\top, \bot, \vdash)| = n$ and $|R_n(\top, \bot, \vdash, \dashv) = 2$

Proofs: Construction of rectangulations



Observation 5: $|R_n(\top, \bot, \vdash)| = n$ and $|R_n(\top, \bot, \vdash, \dashv) = 2$

Proofs: Construction of rectangulations



Summary

	Weak Equivalence	Strong Equivalence
$\top$		
$\top, \bot$		
$\top, \vdash$		
$\top, \bot, \vdash$		
$\top, \bot, \vdash, \dashv$		

Summary

	Weak Equivalence	Strong Equivalence
$\top$	$\lvert R_n^w(\top)\rvert = C_n$	$\lvert R_n^s(\top)\rvert = \lvert I_n(110, 210, 010, 120)\rvert$
$\top, \bot$		
$\top, \vdash$		
$\top, \bot, \vdash$		
$\top, \bot, \vdash, \dashv$		

Summary

	Weak Equivalence	Strong Equivalence
$\top$	$\lvert R_n^w(\top)\rvert = C_n$	$\lvert R_n^s(\top)\rvert = \lvert I_n(110, 210, 010, 120)\rvert$
$\top, \bot$	$\lvert R_n^w(\top, \bot)\rvert = 2^{n-1}$	Bijection to rushed Dyck paths
$\top, \vdash$		
$\top, \bot, \vdash$		
$\top, \bot, \vdash, \dashv$		

Summary

	Weak Equivalence	Strong Equivalence
$\top$	$\lvert R_n^w(\top)\rvert = C_n$	$\lvert R_n^s(\top)\rvert = \lvert I_n(110, 210, 010, 120)\rvert$
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$\top, \vdash$	$\lvert R_n(\top, \vdash)\rvert = 2^{n-1}$	
$\top, \bot, \vdash$		
$\top, \bot, \vdash, \dashv$		

Summary

	Weak Equivalence	Strong Equivalence
$\top$	$\lvert R_n^w(\top)\rvert = C_n$	$\lvert R_n^s(\top)\rvert = \lvert I_n(110, 210, 010, 120)\rvert$
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$\top, \bot, \vdash$	$\lvert R_n(\top, \bot, \vdash)\rvert = n$	
$\top, \bot, \vdash, \dashv$	$\lvert R_n(\top, \bot, \vdash, \dashv)\rvert = 2$	

Summary

	Weak Equivalence	Strong Equivalence
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$\top, \vdash$	$\lvert R_n(\top, \vdash)\rvert = 2^{n-1}$	
$\top, \bot, \vdash$	$\lvert R_n(\top, \bot, \vdash)\rvert = n$	
$\top, \bot, \vdash, \dashv$	$\lvert R_n(\top, \bot, \vdash, \dashv)\rvert = 2$	

THANK YOU!