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# The 334-triangle graph of $\mathrm{SL}_3(\mathbb{Z})$

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Long, Reid, and Thistlethwaite have shown that some groups generated by representations of the  $\Delta 334$  triangle group in  $\mathrm{SL}_3(\mathbb{Z})$  are thin, while the status of others is unknown. We take a new approach: For each group we introduce a new graph that captures information about representations of  $\Delta 334$  in the group. We provide examples of our graph for a variety of groups, and we use information about the graph for  $\mathrm{SL}_3(\mathbb{Z}/2\mathbb{Z})$  to show that the chromatic number of the graph for  $\mathrm{SL}_3(\mathbb{Z})$  is at most 8. By generating a portion of the graph for  $\mathrm{SL}_3(\mathbb{Z})$  we show its chromatic number is at least 4; we conjecture it is equal to 4.

## 1. Introduction

Consider a subgroup  $G$  of  $\mathrm{SL}_n(\mathbb{Z})$ , the group of all  $n \times n$  matrices with integer entries and a determinant of 1, under matrix multiplication. Later we will also consider subgroups of  $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$ , the groups of all  $n \times n$  matrices with determinant 1 and entries in  $\mathbb{Z}/p\mathbb{Z}$ , under matrix multiplication.

We say  $G$  is a thin group whenever  $G$  has infinite index in  $\mathrm{SL}_n(\mathbb{Z})$  and the Zariski closure of  $G$  is all of  $\mathrm{SL}_n(\mathbb{Z})$ . To define the Zariski closure of  $G$ , suppose

$$A = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}$$

is in  $G$  and let  $p$  be a polynomial in the  $n^2$  variables  $x_{11}, x_{12}, \dots, x_{1n}, \dots, x_{n1}, x_{n2}$ , and  $x_{nn}$ . We define  $p(A)$  to be  $p$  evaluated at the entries of  $A$ . Let  $I(G)$  be the set of all polynomials  $p$  such that  $p(A) = 0$  for all  $A \in G$ . The Zariski closure of  $G$  is the set of all matrices  $B \in \mathrm{SL}_n(\mathbb{Z})$  such that for each polynomial  $p \in I(G)$  we have  $p(B) = 0$ . By construction,  $G$  is a subset of its Zariski closure.

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While groups that have both infinite index and are Zariski dense have been studied for the past 150 years, and they have been called thin for the last 15, there are still many groups for which we cannot definitively determine whether or not they are thin [Kontorovich et al. 2019]. We explore this in three (non-)examples.

Consider the subgroup  $G$  of  $\mathrm{SL}_2(\mathbb{Z})$  generated by

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As Kontorovich, Long, Lubotzky, and Reid [Kontorovich et al. 2019] note, it is well known that this group is all of  $\mathrm{SL}_2(\mathbb{Z})$ , and thus has index 1 and is not thin.

Next consider the subgroup  $G$  of  $\mathrm{SL}_3(\mathbb{Z})$  generated by

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -1 \\ 0 & 3 & 1 \end{pmatrix}.$$

We can check that  $G$  is a representation of the  $\Delta 334$  triangle group, which is defined by

$$T = \langle a, b \mid a^3 = b^3 = (ab)^4 = e \rangle. \quad (1)$$

In fact, this representation is faithful; one can use work of [Margulis 1991] to show that this implies  $G$  has infinite index in  $\mathrm{SL}_3(\mathbb{Z})$ . It turns out that the Zariski closure of  $G$  is  $\mathrm{SL}_3(\mathbb{Z})$ , so  $G$  is thin.

By contrast, it is not always known whether a given subgroup of  $\mathrm{SL}_3(\mathbb{Z})$  is thin. For example, consider the group  $G$  generated by

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -3 & -2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -2 & 0 & -1 \\ -5 & 1 & -1 \\ 3 & 0 & 1 \end{pmatrix}.$$

We can mod out by a prime number to show that this group is Zariski dense, as described in [Kontorovich et al. 2019], but it is not known whether the group has infinite index in  $\mathrm{SL}_3(\mathbb{Z})$ .

Many of the thin group candidates studied in [Long et al. 2011], as well as the examples above, are representations of the  $\Delta 334$  triangle group  $T$  defined in (1). In fact, we have used ideas from [Long et al. 2011] to generate thousands of additional thin group candidates, all of which are also representations of  $T$ . In this paper we introduce and study a natural graph on the set of elements of order 3 in a group  $G$  that captures interesting information about the set of representations of  $\Delta 334$  in  $G$ .

For any group  $G$ , let  $\Delta 334(G)$  be the graph whose vertices are the elements  $a \in G$  such that  $a^3 = e$ , in which there is an edge between two vertices  $a$  and  $b$  if and only if  $(ab)^4 = e$ . We note that since  $(ab)^4 = e$  defines a symmetric relation,  $\Delta 334(G)$  is an undirected graph. We call  $\Delta 334(G)$  the *334-triangle graph* of  $G$ .

In this paper we explore the properties of  $\Delta 334(G)$  generally before considering a number of finite examples. We then turn our attention to  $\Delta 334(SL_3(\mathbb{Z}))$ .

Although we do not know of an explicit connection between  $\Delta 334(SL_3(\mathbb{Z}))$  and thin groups in  $SL_3(\mathbb{Z})$ , this graph seems to be of independent interest. Among the properties of  $\Delta 334(SL_3(\mathbb{Z}))$  we could study, we focus on the chromatic number. We use the natural homomorphism from  $SL_3(\mathbb{Z})$  to  $SL_3(\mathbb{Z}/2\mathbb{Z})$  to show that this chromatic number is at most 8, and we examine a small portion of the graph to show it is at least 4. We conjecture it is equal to 4.

## 2. The 334-triangle graph

In this section, we analyze the 334-triangle graph for a variety of groups. To start, we prove four facts about  $\Delta 334(G)$ : the identity element is adjacent to itself and nothing else, the identity element is the only element that is adjacent to itself, every element is adjacent to its inverse, and for abelian groups these are the only edges. We also describe  $\Delta 334(G)$  when  $G$  is a direct product of two groups.

**Lemma 2.1.** *For any group  $G$ , the identity in  $G$  is adjacent in  $\Delta 334(G)$  to itself and is adjacent to no other vertex in  $\Delta 334(G)$ . Furthermore, the identity is the only element in  $\Delta 334(G)$  that is adjacent to itself.*

*Proof.* Let  $G$  be a group and let  $e$  be the identity element of  $G$ . Then  $e^3 = e$  and  $(ee)^4 = e$ , so  $e$  is a vertex in  $\Delta 334(G)$  and is adjacent to itself.

Let  $A$  be an order-3 element of  $G$ , so that  $A^3 = e$ , but  $A \neq e$ . Then,  $(Ae)^4 = A$ , so there is no edge connecting  $A$  to  $e$ . Hence,  $e$  is not adjacent to any other element in the  $\Delta 334$  graph of  $G$ .

Finally, suppose  $A$  is adjacent to itself. Since  $A$  is a vertex in  $\Delta 334(G)$ , we have  $A^3 = e$  and therefore  $A^9 = e$ . Since  $A$  is adjacent to itself, we also have  $(A^2)^4 = e$ , so  $A^8 = e$ . Since  $A^9 = A^8$ , we must have  $A = e$ .  $\square$

In view of Lemma 2.1, we will almost always disregard the identity vertex in further discussions of  $\Delta 334(G)$  and focus only on the nonidentity component(s).

**Lemma 2.2.** *For any group  $G$ , and any vertex  $A$  in  $\Delta 334(G)$ , the element  $A^{-1}$  is also a vertex in  $\Delta 334(G)$ , and  $A$  and  $A^{-1}$  are adjacent.*

*Proof.* Let  $G$  be a group and let  $A \in G$  be an element of  $G$  such that  $A^3 = e$ . If  $A = e$ , then  $A = A^{-1}$ , and by Lemma 2.1,  $A$  is adjacent to itself. Therefore,  $A$  and  $A^{-1}$  are both vertices in  $\Delta 334(G)$  and are adjacent.

Now suppose  $A \neq e$ . Since  $|A| = |A^{-1}|$  and  $A^3 = e$ , we also know that  $(A^{-1})^3 = e$ . Therefore,  $A^{-1}$  is a vertex in  $\Delta 334(G)$ . In addition,  $(AA^{-1})^4 = e^4 = e$ , so  $A$  and  $A^{-1}$  are adjacent.  $\square$

Using Lemmas 2.1 and 2.2, we are able to describe  $\Delta 334(G)$  completely when  $G$  is abelian.

**Lemma 2.3.** *For any abelian group  $G$ , two vertices  $A, B$  in  $\Delta 334(G)$  are adjacent if and only if  $A^{-1} = B$ .*

*Proof.* Let  $G$  be an abelian group and let  $A$  and  $B$  be two elements of  $G$  such that  $A^3 = B^3 = e$ . We know  $A$  and  $B$  will be adjacent if and only if  $(AB)^4 = e$ . However,  $(AB)^4 = A^4B^4 = AB$ . Thus,  $A$  and  $B$  will be adjacent if and only if  $AB = e$ , and this is only true when  $A^{-1} = B$ .  $\square$

We know that in some cases the converse of Lemma 2.3 can fail. That is, there are some nonabelian groups that also have the property that two vertices  $A, B$  are adjacent if and only if  $A^{-1} = B$ . For example, this will happen if  $G$  is not abelian and has odd order divisible by 3. It is not known whether the converse holds for groups whose order is divisible by 6 and that are generated by their order-2 and order-3 elements.

In the case where  $H \subseteq G$ , it turns out that  $\Delta 334(H)$  is a subgraph of  $\Delta 334(G)$ . That is,  $\Delta 334(H)$  is a subset of the vertices in  $\Delta 334(G)$  along with all edges connecting them. Another interesting question is for  $H \triangleleft G$ : how are  $\Delta 334(G)$ ,  $\Delta 334(H)$  and  $\Delta 334(G/H)$  related? This is a question for further research.

Finally, we also consider the 334-triangle graph of a direct sum of groups, showing that the graph  $\Delta 334(G \oplus H)$  is the Kronecker product of  $\Delta 334(G)$  and  $\Delta 334(H)$ .

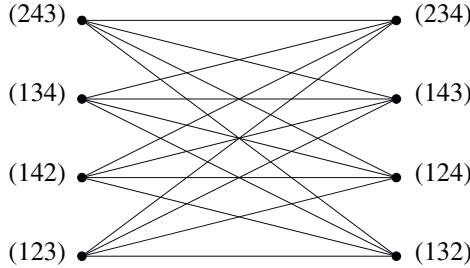
**Definition 2.4.** For any graphs  $G$  and  $H$  with vertices  $g, g' \in G$  and  $h, h' \in H$ , the Kronecker product of  $G$  and  $H$ , written  $G \otimes H$ , is the graph with vertices of the form  $(g, h)$  and in which  $(g, h)$  and  $(g', h')$  are adjacent if and only if both  $g$  and  $g'$  are adjacent in  $G$  and  $h$  and  $h'$  are adjacent in  $H$ .

**Lemma 2.5.** *For any groups  $G$  and  $H$ , we have*

$$\Delta 334(G \oplus H) = \Delta 334(G) \otimes \Delta 334(H).$$

*Proof.* Let  $G$  be a group with elements  $g$  and  $g'$  such that  $g^3 = g'^3 = e_G$  and let  $H$  be a group with elements  $h$  and  $h'$  such that  $h^3 = h'^3 = e_H$ , where  $e_G$  and  $e_H$  are the identity elements of  $G$  and  $H$ , respectively. Consider elements  $(g, h)$  and  $(g', h')$  in  $G \oplus H$ . We wish to show that  $(g, h)$  and  $(g', h')$  will be adjacent in  $\Delta 334(G \oplus H)$  if and only if  $g$  and  $g'$  are adjacent in  $\Delta 334(G)$  and  $h$  and  $h'$  are adjacent in the  $\Delta 334(H)$ .

In order for an element  $(g, h)$  in  $G \oplus H$  to have order 3, both  $g$  and  $h$  must have either order 3 or 1. Thus,  $g$  and  $h$  will both appear in  $\Delta 334(G)$  and  $\Delta 334(H)$ , respectively. If  $g$  and  $g'$  are adjacent in  $\Delta 334(G)$  and  $h$  and  $h'$  are adjacent in  $\Delta 334(H)$ , then  $((g, h)(g', h'))^4 = ((gg')^4, (hh')^4) = (e_G, e_H)$ , so  $(g, h)$  and  $(g', h')$  are adjacent in  $\Delta 334(G \oplus H)$ .



**Figure 1.** The nonidentity component of  $\Delta 334(A_4)$ .

On the other hand, let  $(g, h)$  and  $(g', h')$  be adjacent in  $\Delta 334(G \otimes H)$ . Then,  $((g, h)(g', h'))^4 = ((gg')^4, (hh')^4) = (e_G, e_H)$ , so  $(gg')^4 = e_G$  and  $(hh')^4 = e_H$ , so  $g$  and  $g'$  are adjacent in  $\Delta 334(G)$  and  $h$  and  $h'$  are adjacent in  $\Delta 334(H)$ .  $\square$

Having now laid the groundwork for understanding these graphs, and having completely described the graph for all abelian groups and groups that are isomorphic to direct sums of two or more groups, let us consider  $\Delta 334(G)$  of some groups that are neither of these. We begin by analyzing  $S_4$ , the set of permutations on four letters.  $S_4$  has eight order-3 elements:  $(123)$ ,  $(132)$ ,  $(124)$ ,  $(142)$ ,  $(134)$ ,  $(143)$ ,  $(234)$ , and  $(243)$ . Thus, the nonidentity component of the graph will have eight vertices. We are interested in the subgroup these elements generate, since any element of the group that is not part of our subgroup will not appear in the graph. For  $S_4$ , the relevant subgroup is  $A_4$ . Therefore, the graphs for  $S_4$  and  $A_4$  are the same, so we will focus our discussion on  $A_4$ . (Indeed, the same reasoning shows  $\Delta 334(S_n) = \Delta 334(A_n)$  for all  $n \geq 2$ .)

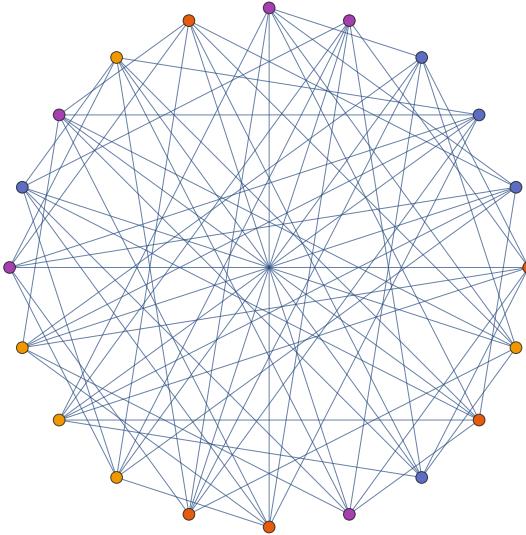
We can check that the order-3 elements of  $A_4$  fall into two conjugacy classes, each with four elements. As we will show, each order-3 element is adjacent to exactly those order-3 elements to which it is not conjugate. To see this, note that for two order-3 elements in  $S_4$  there are two possibilities for their product:

$$(abc)(abd) = (ac)(bd), \quad (2)$$

$$(abc)(bad) = (adc). \quad (3)$$

Since these are the only two options, and  $(adc)^4 \neq e$ , there will be an edge connecting elements  $A$  and  $B$  if and only if  $A$  is of the form  $(abc)$  and  $B$  is of the form  $(abd)$ . We can check that this occurs exactly when  $A$  and  $B$  are not conjugate in  $A_4$ .

Based on the analysis above, we find that the graph  $\Delta 334(A_4)$  is as in Figure 1. This is a complete bipartite graph, with the conjugacy classes forming the bipartition. This graph has chromatic number 2 and has cycles of length 4, 6, and 8; its clique number is 2.



**Figure 2.** The nonidentity component of  $\Delta 334(S_5)$ .

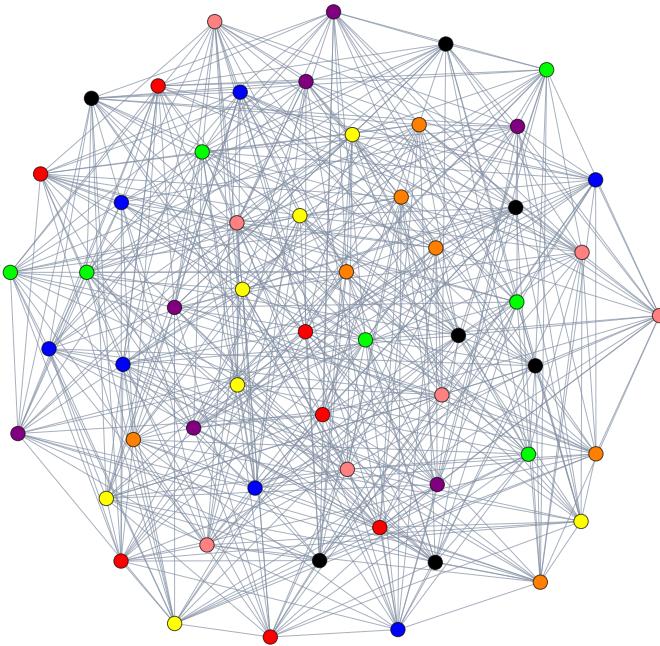
We leave it to the reader to verify the interesting fact that  $SL_2(\mathbb{Z}/3\mathbb{Z})$  and  $S_4$ , which have the same order, also have isomorphic 334-triangle graphs.

As another example, we observe that the 334-triangle graph of  $S_5$  (and therefore of  $A_5$ ) is not bipartite — (125), (124), and (123) form a three-cycle — and every vertex has degree 7. This graph is shown in Figure 2.

We are most interested in  $\Delta 334(SL_3(\mathbb{Z}))$ , since we have generated many candidates for thin groups that are subgroups of  $SL_3(\mathbb{Z})$ , as mentioned in Section 1. However, this graph is infinite, so we will begin with finite graphs that are images of this graph after modding out by a prime. First we analyze the nonidentity component of  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$ , which is shown in Figure 3. We used Mathematica to generate all of the order-3 elements of  $SL_3(\mathbb{Z}/2\mathbb{Z})$ . There are 56 such elements; thus, the nonidentity component of  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$  contains 56 vertices. Every vertex has degree 19 and there are cycles of all lengths from 3 to 56. In particular, this component is connected and Hamiltonian. It also has chromatic number 8 and cliques up to size 5. One such clique is

$$\left\{ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}.$$

We are able to say less about  $\Delta 334(SL_3(\mathbb{Z}/3\mathbb{Z}))$  because the graph becomes so much larger. We know there are 728 matrices of order 3 in  $SL_3(\mathbb{Z}/3\mathbb{Z})$ , so the nonidentity component of  $\Delta 334(SL_3(\mathbb{Z}/3\mathbb{Z}))$  will contain 728 vertices. We know



**Figure 3.** The nonidentity component of  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$ .

there is a single connected nonidentity component. All vertices have degree either 118 or 136. We were unable to determine the length of cycles or chromatic number of this graph due to its size.

### 3. $\Delta 334(SL_3(\mathbb{Z}))$

We now consider  $\Delta 334(SL_3(\mathbb{Z}))$ . We know  $SL_3(\mathbb{Z})$  is infinite, and we can show it has an infinite number of order-3 elements. For example, for any  $a, b, c \in \mathbb{Z}$ ,

$$\begin{pmatrix} 1 & 3a & 3b \\ 0 & -2-3c & -1-3c-3c^2 \\ 0 & 3 & 1+3c \end{pmatrix}$$

has integer entries, determinant 1, and order 3. Thus,  $\Delta 334(SL_3(\mathbb{Z}))$  is also infinite. However, using what we know about  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$  we are able to put bounds on the chromatic number of  $\Delta 334(SL_3(\mathbb{Z}))$ . We will do this by showing that any edge in  $\Delta 334(SL_3(\mathbb{Z}))$  reduces to an edge in  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$ . We will then lift a proper coloring from  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$  to  $\Delta 334(SL_3(\mathbb{Z}))$  by coloring each vertex in  $\Delta 334(SL_3(\mathbb{Z}))$  the same color as its image in  $\Delta 334(SL_3(\mathbb{Z}/2\mathbb{Z}))$ .

Before we begin, note that we are using the fact that the natural homomorphism from  $SL_3(\mathbb{Z})$  to  $SL_3(\mathbb{Z}/2\mathbb{Z})$  induces a graph homomorphism from  $\Delta 334(SL_3(\mathbb{Z}))$

to  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/2\mathbb{Z}))$ . It's worth noting that in general any homomorphism from a group  $G$  to a group  $H$  will similarly induce a graph homomorphism from  $\Delta 334(G)$  to  $\Delta 334(H)$ , underscoring the fact that the 334-triangle graph is natural from an algebraic point of view.

We first show that no vertex in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  reduces to the identity modulo any prime.

**Lemma 3.1.** *For any matrix  $A \in \mathrm{SL}_3(\mathbb{Z})$  with order 3,  $A$  cannot be equivalent to the identity matrix modulo any prime.*

*Proof.* Let  $A$  be a matrix in  $\mathrm{SL}_3(\mathbb{Z})$  with  $A^3 = I_3$ , where  $I_3$  is the identity element of  $\mathrm{SL}_3(\mathbb{Z})$ , and assume  $A \neq I_3$ . Let  $p$  be prime, and assume by way of contradiction that  $A$  is equivalent to the identity modulo  $p$ .

Since  $A^3 = I_3$ , the minimal polynomial for  $A$  over  $\mathbb{Q}$  must divide  $x^3 - 1 = (x - 1)(x^2 + x + 1)$ . Since  $A \neq I_3$ , the minimal polynomial cannot be  $x - 1$ . We also note that  $x^2 + x + 1 = (x - \omega)(x - \bar{\omega})$ , where  $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$  is a primitive third root of unity. Since  $\omega$  and  $\bar{\omega}$  are not rational, the minimal polynomial must be  $x^2 + x + 1$  or  $x^3 - 1$ . But if the minimal polynomial were  $x^2 + x + 1$  then by the Cayley–Hamilton theorem the characteristic polynomial would be  $-(x - \omega)(x^2 + x + 1)$  or  $-(x - \bar{\omega})(x^2 + x + 1)$ , neither of which has constant term equal to  $\det A = 1$ . Therefore, the minimal polynomial for  $A$  over  $\mathbb{Q}$  is  $x^3 - 1$ .

The fact that the minimal polynomial for  $A$  over  $\mathbb{Q}$  is  $x^3 - 1$  implies  $A$  has an eigenvector  $\vec{v}_1$  with eigenvalue 1, and that there is a vector  $\vec{v}_2 \neq \vec{0}$  with  $(A^2 + A + I_3)\vec{v}_2 = \vec{0}$ . We set  $\vec{v}_3 = A\vec{v}_2$ . By scaling if necessary, we can assume all of the entries of  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  are integers and that the entries of  $\vec{v}_2$  have no common prime factor. We can check that  $A\vec{v}_3 = -\vec{v}_2 - \vec{v}_3$  and that  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$  form a basis for  $\mathbb{Q}^3$ . Therefore, there is an invertible  $3 \times 3$  matrix  $M$  with entries in  $\mathbb{Z}$  such that

$$AM = M \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

In particular, the columns of  $M$  are  $\vec{v}_1$ ,  $\vec{v}_2$ , and  $\vec{v}_3$ . We can now use our assumption that  $A$  is equivalent to the identity modulo  $p$  to check that the entries of the middle column of  $M$  are all divisible by  $p$ . But this contradicts the fact that the entries of  $\vec{v}_2$  do not have a common prime factor.  $\square$

Having shown that no matrices in the nonidentity component of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  reduce to the identity modulo a prime  $p$ , we can show that every edge in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  maps to an edge in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ .

**Lemma 3.2.** *Let  $A$  and  $B$  with  $A \neq B$  be adjacent vertices in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ . Then, for  $A' = A \bmod p$  and  $B' = B \bmod p$ ,  $A'$  and  $B'$  are adjacent in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$  and  $A' \neq B'$ .*

*Proof.* Let  $A, B$  be adjacent vertices in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  with  $A \neq B$  and let  $p$  be prime. By Lemma 2.1, neither  $A$  nor  $B$  are the identity, and by Lemma 3.1 neither reduce to the identity modulo  $p$ .

Let  $C = AB$  and let  $C' = A'B'$ . We know that  $C' = C \pmod{p}$  by the rules of modular arithmetic. We know that  $C^4 = I_3$ , so  $C'^4$  must also equal the identity. Thus,  $A'$  and  $B'$  are adjacent in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ . By Lemma 2.1, the only loop in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$  is at the identity and  $A \neq B \neq I_3$ , so  $A' \neq B'$ .  $\square$

Finally, since each edge in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  maps to an edge in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ , we are able to set an upper bound on the chromatic number,  $\chi(\Delta 334(\mathrm{SL}_3(\mathbb{Z})))$ .

**Theorem 3.3.** *For any prime  $p$ , we have*

$$\chi(\Delta 334(\mathrm{SL}_3(\mathbb{Z}))) \leq \chi(\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))).$$

*Proof.* If we have a proper coloring of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$  where  $p$  is prime, then we can lift it to a proper coloring of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ . We do this by coloring every vertex in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  the same color as its image in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ . For any two adjacent vertices in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ , their images are also adjacent in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ , so they will have different colors. Thus, we will have a proper coloring of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ . Therefore, the chromatic number of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  is at most the chromatic number of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}/p\mathbb{Z}))$ .  $\square$

Since  $\chi(\Delta 334(\mathrm{SL}_3(\mathbb{Z}/2\mathbb{Z}))) = 8$ , and 2 is prime, by Theorem 3.3 we know that the chromatic number of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  is at most 8. We have used Mathematica to generate a finite portion of this graph with about 25,000 vertices; the chromatic number of this portion is 4. Thus, one lower bound for the chromatic number of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  is 4. We conjecture that  $\chi(\Delta 334(\mathrm{SL}_3(\mathbb{Z})))$  is exactly 4.

**Conjecture 3.4.**  $\chi(\Delta 334(\mathrm{SL}_3(\mathbb{Z}))) = 4$ .

Using the IGraph/M package, which uses the Boyer–Myrvold algorithm, we have found that the nonidentity part of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  that we have generated so far is nonplanar. Thus, the overall graph is also nonplanar. Additionally, it is connected, has cycles of varying lengths, and has no cliques of size greater than 3. We conjecture that all three of these facts hold for the entire graph  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ .

**Conjecture 3.5.** *The nonidentity component of  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$  is connected.*

**Conjecture 3.6.** *There are cycles of every length in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ .*

**Conjecture 3.7.** *There are no cliques of size greater than 3 in  $\Delta 334(\mathrm{SL}_3(\mathbb{Z}))$ .*

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